The Wave Equation

The wave equation with Dirichlet boundary conditions models vibrations in a string. The function $u(x,t)$ gives the vertical displacement of the string at location x and time t.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$u(0,t) = u(l,t) = 0$$

$$u(x,0) = \psi(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \gamma(x)$$

Because this is a second order equation in the t variable, we need two sets of initial conditions for t. The first set gives the initial displacement of the string, while the second set gives the initial vertical velocity of the string.

A road map to the solution

In these notes I am going to develop the d'Alembert solution for the wave equation. This solution is actually a combination of three solution techniques, each of which is interesting in its own right.

The first technique is an operator factorization technique. This technique will make it possible for us to write the general solution to the wave equation, u_g as the combination of two solutions to simpler problems:

$$u_g = u_1 + u_2$$

In the course of solving the two subproblems whose solutions are u_1 and u_2 we will meet a second technique, the *method of characteristics*.

To fully resolve the general solution into a specific solution that meets the boundary conditions for the wave equation, we will make use of a superposition technique. We are going to use the general solution to first derive solutions to two simpler problems. The function u_a will solve the problem

$$u_a(0,t) = u_a(l,t) = 0$$

$$u_a(x,0) = \psi(x)$$

$$\frac{\partial u_a}{\partial t}(x,0) = 0$$

while the function u_b will solve the problem

$$u_b(0,t) = u_b(l,t) = 0$$

$$u_b(x,0) = 0$$

$$\frac{\partial u_b}{\partial t}(x,0) = \gamma(x)$$

By superposition the solution to the original boundary conditions will then be

$$u(x,t) = u_a(x,t) + u_b(x,t)$$

Factorization

On occasion, it will be possible to take a linear, constant coefficient, second order operator and factor as the combination of two first order constant coefficient operators. In the case of the wave equation, the operator in question

$$L\,u = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}$$

can be factored into two first order operators.

$$L\,u = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u(x,t) = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u(x,t)$$

What is going on here is a specific example of the factorization technique. In this technique we start with an operator that can be factored into a pair of linear operators that commute.

$$L\,u = L_1\,L_2\,u = L_2\,L_1\,u$$

In looking for solutions to the equation

$$L\,u = 0$$

we start by seeking functions u_1 and u_2 that satisfy

$$L_1\,u_1 = 0$$

$$L_2\,u_2 = 0$$

Since these are both first order PDEs, these equations will be easier to solve than the original equation. Once we are able to solve these two equations, we note that

$$L\left(c_1\, u_1 + c_2\, u_2\right) = \left(L\, c_1\, u_1\right) + \left(L\, c_2\, u_2\right)$$

$$= c_1\, L_2\, L_1\, u_1 + c_2\, L_1\, L_2\, u_2$$

$$= c_1\, L_2\, 0 + c_2\, L_1\, 0 = 0$$

The choice of c_1 and c_2 in this solution is somewhat arbitrary, in that any choice the we make now will get absorbed into the functions u_1 and u_2 when we apply the boundary conditions later. For that reason I will pick the simplest combination, $c_1 = c_2 = 1$. With this choice, the general solution to the wave equation before boundary conditions becomes

$$u_g(x,t) = u_1(x,t) + u_2(x,t)$$

The method of characteristics

To solve the subproblems for u_1 and u_2 we will apply is the *method of characteristics*, which is a method that is frequently used to solve first order, constant coefficient PDEs. Consider the first subproblem:

$$L_1\, u_1 = \left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right) u(x,t) = 0$$

To find a solution to this equation, we seek a change of variables that allows us to write both t and x as functions of two new variables p and q. If we fix $q = q_0$ and allow p to vary, $x(p,q_0)$ and $t(p,q_0)$ will trace out curves in space known as *characteristic curves*.

Characteristic curves are constructed by making two key demands:

1. Curves of constant q should correspond to curve of constant u_1.
2. The *initial curve*, $t = 0$, should map to a curve $p = 0$. Everywhere along that curve we should have $q = x$.

We can meet the first requirement by demanding

$$\frac{\partial}{\partial p}\, u(x(p,q_0),t(p,q_0)) = \left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right) u(x,t) = 0$$

If we can do this, we will have transformed our original equation into an equation

$$\frac{\partial}{\partial p}\, u(p,q_0) = 0$$

which is much easier to solve. To determine what the parameterization needs to be, we apply the chain rule for partial differentiation:

$$\frac{\partial}{\partial p} u(x(p,q_0),t(p,q_0)) = \frac{\partial x}{\partial p}\frac{\partial}{\partial x} u(x,t) + \frac{\partial t}{\partial p}\frac{\partial}{\partial t} u(x,t) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(x,t)$$

This forces

$$\frac{\partial x}{\partial p} = -c$$

$$\frac{\partial t}{\partial p} = 1$$

which has solutions

$$x = -c\,p + c_1(q)$$

$$t = p + c_2(q)$$

The second requirement above then forces

$$c_2(q) = 0$$

$$c_1(q) = x = q$$

This now gives us the desired transformation formulas:

$$t(p,q) = p$$

$$x(p,q) = -c\,p + q$$

This change of variables can also be easily inverted:

$$p(x,t) = t$$

$$q(x,t) = x + c\,t$$

Now we return to the original problem for $u_1(x,t)$:

$$L_1\,u_1 = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u_1(x,t) = 0$$

To determine what these equations turn into we first do some applications of the chain rule:

$$\frac{\partial}{\partial t}u_1(x,t) = \frac{\partial p}{\partial t}\frac{\partial}{\partial p}u_1(p,q) + \frac{\partial q}{\partial t}\frac{\partial}{\partial q}u_1(p,q) = \frac{\partial}{\partial p}u_1(p,q) + c\,\frac{\partial}{\partial q}u_1(p,q)$$

$$\frac{\partial}{\partial x}u_1(x,t) = \frac{\partial p}{\partial x}\frac{\partial}{\partial p}u_1(p,q) + \frac{\partial q}{\partial x}\frac{\partial}{\partial q}u_1(p,q) = 0 + \frac{\partial}{\partial q}u_1(p,q)$$

This now turns the equation above into

$$\left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right)u_1(x,t) = \frac{\partial}{\partial p}u_1(p,q) + c\,\frac{\partial}{\partial q}u_1(p,q) - c\,\frac{\partial}{\partial q}u_1(p,q) = \frac{\partial}{\partial p}u_1(p,q) = 0$$

as desired.

This equation has solution

$$u_1(p,q) = f(q) = f(x+c\,t) = u_1(x,t)$$

We will determine the unknown function below when we apply boundary conditions.

To solve the second equation

$$L_2\,u_2 = 0$$

we use a similar approach. I will spare you the details, but that equation can also be solved by a change of variables

$$t(p,q) = p$$

$$x(p,q) = c\,p + q$$

This change of variables can also be easily inverted:

$$p(x,t) = t$$

$$q(x,t) = x - c\,t$$

This change of variables converts

$$L_2\,u_2 = 0$$

into

$$\frac{\partial}{\partial p}u_2(p,q) = 0$$

which has solution

$$u_2(p,q) = g(q) = g(x-c\,t) = u_2(x,t)$$

This now gives us the general solution to the original problem:

$$u_g(x,t) = u_1(x,t) + u_2(x,t) = f(x+c\,t) + g(x-c\,t)$$

Applying the boundary conditions

To apply the boundary conditions for the wave equation

$$u(x,0) = u_1(x,0) + u_2(x,0) = \psi(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{\partial u_1}{\partial t}(x,0) + \frac{\partial u_2}{\partial t}(x,0)= \gamma(x)$$

we apply a simple trick. We actually solve two separate problems. The boundary conditions for the first problem are

$$u_a(x,0) = u_1(x,0) + u_2(x,0) = \psi(x)$$

$$\frac{\partial u_a}{\partial t}(x,0) = \frac{\partial u_1}{\partial t}(x,0) + \frac{\partial u_2}{\partial t}(x,0)= 0$$

The boundary conditions for the second problem are

$$u_b(x,0) = u_1(x,0) + u_2(x,0) = 0$$

$$\frac{\partial u_b}{\partial t}(x,0) = \frac{\partial u_1}{\partial t}(x,0) + \frac{\partial u_2}{\partial t}(x,0)= \gamma(x)$$

The principle of superposition then tells us that the sum of the solutions to these two simpler problems will solve the original set of boundary conditions.

To solve the first problem we assume that the solution looks like

$$f_1(x+c\,t) + g_1(x-c\,t)$$

and apply the boundary conditions to get that

$$u_a(x,0) = f_1(x) + g_1(x) = \psi(x)$$

$$\frac{\partial u_a}{\partial t}(x,0) = -c\,f_1'(x) + c\,g_1'(x) = 0$$

To satisfy the first condition, we can simply pick

$$f_1(x) = g_1(x) = \frac{\psi(x)}{2}$$

This will immediately satisfy the second condition:

$$\frac{\partial u}{\partial t}(x,0) = -c\,f_1'(x) + c\,g_1'(x) = -\,c\,\psi'(x) + c\,\psi'(x) = 0$$

Thus we see that

$$f_1(x+c\,t) + g_1(x-c\,t) = \frac{\psi(x - c\,t)}{2} + \frac{\psi(x + c\,t)}{2}$$

To solve the second problem we need

$$u_b(x,0) = f_2(x) + g_2(x) = 0$$

$$\frac{\partial u_b}{\partial t}(x,0) = -c\,f_2'(x) + c\,g_2'(x) = \gamma(x)$$

The first equation leads immediately to

$$f_2(x) = -g_2(x)$$

Substituting this into the second equation gives

$$\frac{\partial u}{\partial t}(x,0) = -c\,f_2'(x) + c\,g_2'(x) = c\,g_2'(x) + c\,g_2'(x) = \gamma(x)$$

which has solution

$$g_2(x) = \frac{1}{2\,c}\int_0^x \gamma(s)\,\mathrm{d}\,s$$

Thus, the second problem has solution

$$f_2(x - c\,t) + g_2(x + c\,t) = -g_2(x - c\,t) + g_2(x + c\,t)$$

$$= -\frac{1}{2\,c}\int_0^{x-ct} \gamma(s)\,\mathrm{d}\,s + \frac{1}{2\,c}\int_0^{x+ct} \gamma(s)\,\mathrm{d}\,s$$

$$= \frac{1}{2\,c}\int_{x-ct}^{x+ct} \gamma(s)\,\mathrm{d}\,s$$

The complete solution to the problem with the original initial conditions is now just the sum of these two separate solutions:

$$u(x,t) = \frac{\psi(x-ct)}{2} + \frac{\psi(x+ct)}{2} + \frac{1}{2\,c}\int_{x-ct}^{x+ct} \gamma(s)\,\mathrm{d}\,s$$