Heat equation with Neumann boundary conditions

We seek to solve

$$\rho\ c\ \frac{\partial u}{\partial t} - \kappa\ \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0$$

$$u(x,0) = \psi(x)$$

A preliminary problem

To generate some insight into the special behavior of this problem, we start by considering a simpler, related BVP:

$$-\kappa\ \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = f(x)$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}(0) = \frac{\mathrm{d}u}{\mathrm{d}x}(l) = 0$$

To handle this we introduce a differential operator L_N defined on the subspace of functions that satisfy a Neumann boundary condition.

$$L_N(u) = \{ -\kappa\ u'' \mid u \text{ such that } u'(0) = u'(l) = 0 \}$$

The eigenfunctions of this differential operator are $u_0(x) = 1$ and $u_k(x) = \cos(n\ \pi\ x/l)$ for $k = 1, 2, 3,$ ...

As before, we start by multiplying both sides of the ODE by these eigenfunctions and integrating over the interval in question. There are two separate cases to consider. For $k = 0$ we have

$$\frac{1}{l}\int_0^l -\kappa\ \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\,dx = \frac{1}{l}\int_0^l f(x)\,dx$$

The integral on the left simplifies to

$$\frac{1}{l}\int_0^l -\kappa\ \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\,dx = -\frac{\kappa}{l}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\Big|_0^l = 0$$

In general, the integral on the right won't be 0. That means that we are potentially stuck, unless we introduce a *compatibility condition*:

$$\frac{1}{l}\int_0^l f(x)\,dx = 0$$

The reason that we have to impose this extra condition is that the operator $L_N(u)$ has a non-trivial null space. That means that we should expect the equation

$$L_N(u) = f$$

to not have a solution for some functions $f(x)$. The work-around for the moment is to impose this extra condition on $f(x)$ to guarantee that the problem will have a solution.

Assuming that this extra condition is satisfied, we can go to handle the integrals arising from the other eigenfunctions:

$$\frac{2}{l}\int_0^l -\kappa \frac{d^2 u}{dx^2} \cos\left(\frac{k\pi}{l} x\right) dx = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{k\pi}{l} x\right) dx$$

On the left we get after a couple of integrations by parts:

$$\frac{2}{l}\int_0^l -\kappa \frac{d^2 u}{dx^2} \cos\left(\frac{k\pi}{l} x\right) dx$$

$$= -2\frac{\kappa}{l}\frac{du}{dx}\cos\left(\frac{k\pi}{l} x\right)\Big|_0^l - \frac{2\kappa k\pi}{l^2}\int_0^l \frac{du}{dx}\sin\left(\frac{k\pi}{l} x\right) dx$$

$$= 0 - \frac{2\kappa k\pi}{l^2}\left(u(x)\sin\left(\frac{k\pi}{l} x\right)\right)\Big|_0^l + \frac{2\kappa k^2\pi^2}{l^3}\int_0^l u(x)\cos\left(\frac{k\pi}{l} x\right) dx$$

Solving the full problem

We now return to

$$\rho\, c\, \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0$$

$$u(x,0) = \psi(x)$$

Before we seek to solve this, we may want to impose a compatibility condition similar to what we imposed in the earlier problem.

$$\frac{1}{l}\int_0^l f(x,t)\, dx = 0$$

If the $f(x,t)$ we are given does not satisfy this compatibility condition, we can try the following trick. Introduce the function

$$g(t) = \frac{1}{l}\int_0^l f(x,t)\, dx$$

and let $\beta(t)$ be the solution to

$$\rho\; c\; \beta'(t) = g(t)$$

$$\beta(0) = \frac{1}{l} \int_0^l \psi(x)\, dx$$

and instead try to solve

$$\rho\; c\; \frac{\partial u}{\partial t} - \kappa\; \frac{\partial^2 u}{\partial x^2} = f(x,t) - g(t)$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0$$

$$u(x,0) = \psi(x) - \beta(0)$$

By construction, this modified problem will automatically satisfy the compatibility condition:

$$\frac{1}{l} \int_0^l f(x,t) - g(t)\, dx = \frac{1}{l} \int_0^l f(x,t)\, dx - \frac{1}{l} \int_0^l g(t)\, dx$$

$$= g(t) - \frac{1}{l}(l\; g(t)) = 0$$

How does the solution to the modified problem help us to solve the original problem? Let $u_g(x)$ be the solution to

$$\rho\; c\; \frac{\partial u_g}{\partial t} - \kappa\; \frac{\partial^2 u_g}{\partial x^2} = f(x,t) - g(t)$$

$$\frac{\partial u_g}{\partial x}(0,t) = \frac{\partial u_g}{\partial x}(l,t) = 0$$

$$u_g(x,0) = \psi(x) - \beta(0)$$

by linearity of the operator we have that

$$\rho\; c\; \frac{\partial(u_g(x,t)+\beta(t))}{\partial t} - \kappa\; \frac{\partial^2(u_g(x,t)+\beta(t))}{\partial x^2}$$

$$= \left(\rho\; c\; \frac{\partial u_g}{\partial t} - \kappa\; \frac{\partial^2 u_g}{\partial x^2}\right) + \left(\rho\; c\; \frac{\partial(\beta(t))}{\partial t} - \kappa\; \frac{\partial^2(\beta(t))}{\partial x^2}\right)$$

$$= (f(x,t) - g(t)) + g(t) = f(x,t)$$

$$(u_g(x,0) + \beta(0)) = \psi(x) - \beta(0) + \beta(0) = \psi(x)$$

Thus $u_g(x,t)+\beta(t)$ solves the original problem.

The bottom line now is that we have two problems to solve. The simpler problem

$$\rho \; c \; \beta'(t) = g(t)$$

$$\beta(0) = \frac{1}{l} \int_0^l \psi(x) \, dx$$

can be solved immediately by integration:

$$\beta(t) = \int_0^t \frac{g(s)}{\rho \; c} \, ds + \frac{1}{l} \int_0^l \psi(x) \, dx$$

The main problem

$$\rho \; c \, \frac{\partial u_g}{\partial t} - \kappa \, \frac{\partial^2 u_g}{\partial x^2} = f(x,t) - g(t)$$

$$\frac{\partial u_g}{\partial x}(0,t) = \frac{\partial u_g}{\partial x}(l,t) = 0$$

$$u_g(x,0) = \psi(x) - \beta(0)$$

satisfies a compatibility condition, so we can expect to solve it by appropriate application of the Finite element method.