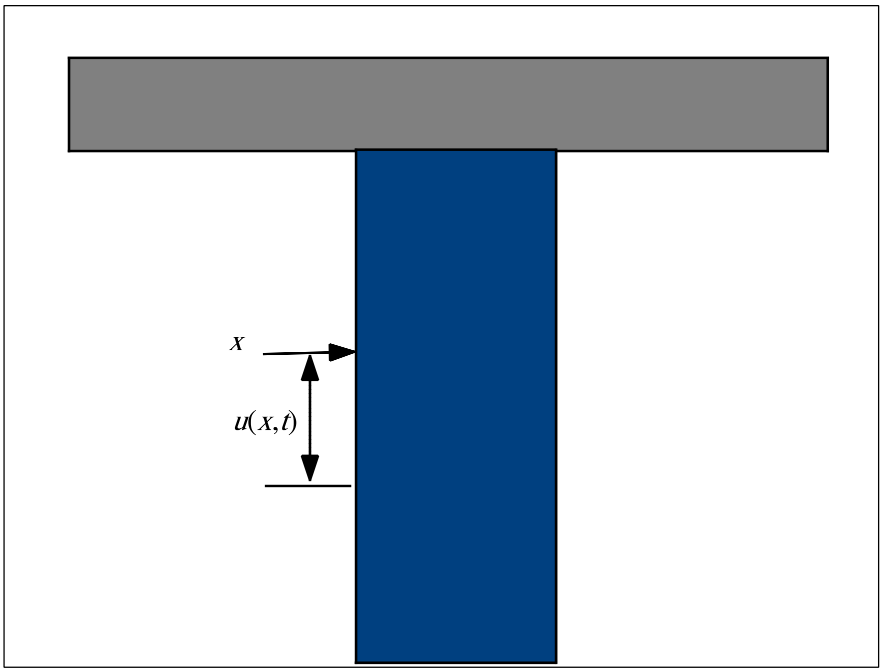
The Wave Equation

Consider a bar made of an elastic material. The bar hangs down vertically from an attachment point $x = 0$ and can vibrate vertically but not horizontally. Since chapter 5 is the chapter on boundary value problems, we will eventually need to manufacture a boundary value problem from this model. The easiest way to do this is to suppose also that the lower end at $x = l$ is fixed in place.

Suppose that we label each point on the bar with its equilibrium location, x, where x is measured vertically from the top.



At any given point in time, the portion of the bar labeled x will be found at a location other than x as the bar vibrates. Let $u(x,t)$ stand for the displacement measured from where the portion of the bar at x started at equilibrium. If that portion of the bar is displaced downward from its equilibrium location, $u(x,t)$ will be positive.

To construct an equation of motion for this system we will use Newton's second law as our starting point:

$$F = m\, a$$

In terms of function $u(x,t)$ introduced above, we have

$$a = \frac{\partial^2 u(x,t)}{\partial t^2}$$

The F will have two components, internal force and external force. The external force (if there is one) will be modeled by a function $f(x,t)$. The internal force is commonly referred to as a *stress* produced as portions of the bar move away from their initial positions. The key to understanding this stress is Young's equation

$$stress = k(x)\ strain$$

where $k(x)$ is the bar's *Young's modulus of elasticity*, which measures the bar's resistance to displacement, and the *strain* term measures the level of stretching or compression in some portion of the bar. $k(x)$ is a function of x because we may be dealing with a bar with non-uniform material properties that vary as we move up and down the bar.

Consider the portion of the bar that was originally located between x and $x + \Delta x$ at equilibrium. As the bar moves, the part that was at x gets displaced to $x + u(x,t)$ and the part that was at $x + \Delta x$ moves to $x + \Delta x + u(x+\Delta x,t)$. From this we can compute the new length of the portion that originally ran from x to $x + u(x,t)$:

$$(x + \Delta x + u(x+\Delta x,t)) - (x + u(x,t)) = \Delta x + (u(x+\Delta x,t)-u(x,t))$$

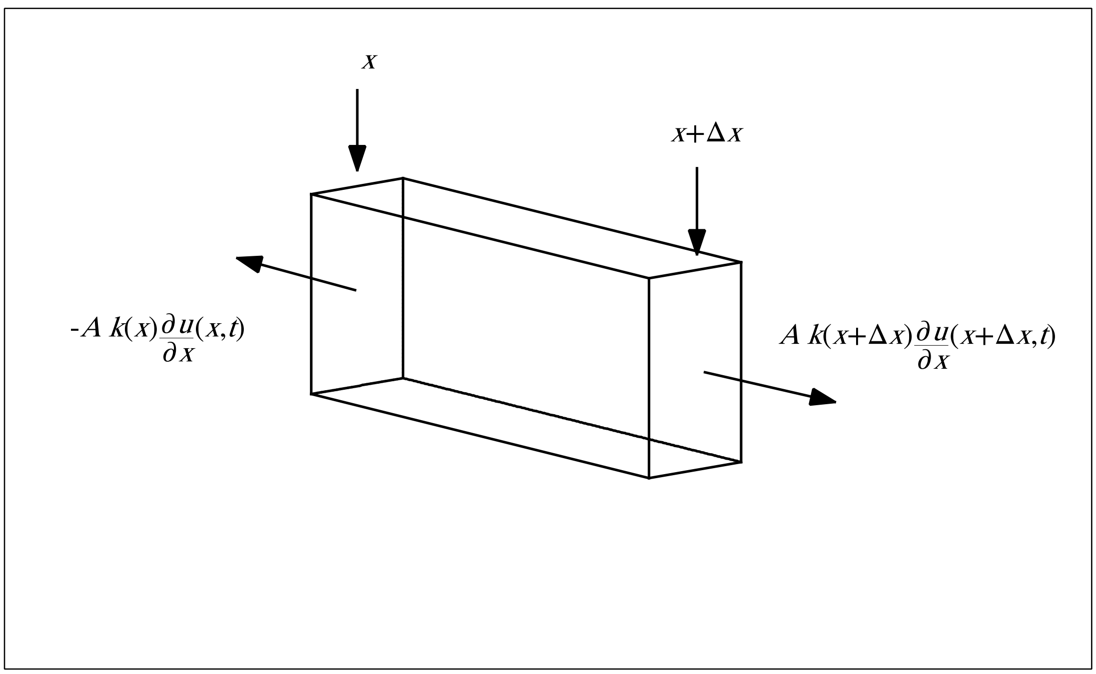
Since the original portion had length Δx, we have experienced a net change in length of $u(x+\Delta x,t) - u(x,t)$. If we average this change over the entire portion we get an average local stretching or compression factor of

$$\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}$$

In the limit, as Δx gets small, this ratio approaches $\frac{\partial u(x,t)}{\partial x}$: this will be our strain term.

The most important thing to know about the stress force is that it is a *net force*: that is, we can only compute the size of the force by observing its effect (the strain) and then reasoning that some net force must have produced that effect.

Consider some portion of the bar that was originally located between x and $x + \Delta x$. Using the reasoning above, we can compute the net stress at both ends of the portion.



Adding these two net stresses at either end gives the net stress force acting on the entire portion of the bar.

$$\textit{net stress} = A\, k(x+\Delta x)\frac{\partial u}{\partial x}(x+\Delta x,t) - A\, k(x)\frac{\partial u}{\partial x}(x,t)$$

Now we introduce a second force. The second force acting on the bar is an external force that gets applied to every part of the bar. To make this sufficiently general, we will assume that the external force $f(x)$ depends on the original location x of whatever part of the bar we are looking at. (Note: a more realistic model would say that the force depends on exactly where that part is at time t, that is, the force should take the form $f(x + u(x,t))$. Since this is too complex to deal with, we simplify things by approximating $f(x + u(x,t))$ with $f(x)$.)

The most important thing to know about this external force is that unlike the stress, which is a net force, the external force is a *bulk force*. This means that to determine its net effect on some portion of the bar we have to integrate this bulk force over every tiny element $A\ dx$ of the bar:

$$\textit{total external force} = \int_x^{x+\Delta x} f(s,t)\, A\, \mathrm{d}\, s$$

At this point we face a small problem. We have two forces acting on the portion of the bar, but they take two different forms mathematically. The external force is expressed as an integral, but the stress is not. Fortunately, this is easy to fix. An application of the first fundamental theorem of calculus gives us

$$A\, k(x+\Delta x)\frac{\partial u}{\partial x}(x+\Delta x,t) - A\, k(x)\frac{\partial u}{\partial x}(x,t) = \int_x^{x+\Delta x} A\, \frac{\partial}{\partial x}\left(k(s)\, \frac{\partial u}{\partial x}(s,t)\right) \mathrm{d}\, s$$

This allows us to also write the stress as an integral at the expense of introducing another derivative with respect to x.

We are now in a position to build a full physical model for the motion of the bar. The key physical principle to use here is Newton's second law of motion, which says that at all points on the bar the net force is equal to the mass times the acceleration.

$$F = m\, a$$

We can easily compute the acceleration by taking the second derivative of the displacement $u(x,t)$ with respect to t. If $\rho(x)$ is the density of the material at x, we have the following.

$$\int_x^{x+\Delta x} A\, \frac{\partial}{\partial x}\left(k(s)\, \frac{\partial u}{\partial x}(s,t)\right) \mathrm{d}\, s + \int_x^{x+\Delta x} f(s,t)\, A\, \mathrm{d}\, s = \int_x^{x+\Delta x} A\, \rho(s)\, \frac{\partial^2 u}{\partial t^2}(s,t)\, \mathrm{d}\, s$$

Rearranging and cancelling the common factor of A found in all the integrals leads to

$$\int_x^{x+\Delta x} \rho(s)\, \frac{\partial^2 u}{\partial t^2}(s,t) - \frac{\partial}{\partial x}\left(k(s)\, \frac{\partial u}{\partial x}(s,t)\right) - f(s,t)\, \mathrm{d}\, s = 0$$

Now observe that this integral has to equal 0 for any choice of subinterval $[x,x+\Delta x]$ in $[0,l]$. A few minute's thought will convince you that the only way to get this integral to vanish for every possible choice of x and Δx is for the integrand to be identically 0 at every x:

$$\rho(x)\, \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial}{\partial x}\left(k(x)\, \frac{\partial u}{\partial x}(x,t)\right) - f(x,t) = 0$$

Note that what we have now is a partial differential equation for the displacement, called the *wave equation*. Most frequently this PDE is written

$$\rho(x)\, \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial}{\partial x}\left(k(x)\, \frac{\partial u}{\partial x}(x,t)\right) = f(x,t)$$

There are also some special forms of this PDE that are worth considering. In the absence of external forces we have the so-called *homogeneous form* of the wave equation.

$$\rho(x)\, \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left(k(x)\, \frac{\partial u(x,t)}{\partial x}\right) = 0$$

Changing the PDE to an ODE

Since the wave equation is a PDE, and we want to restrict our attention to ODEs as long as we are still in chapter five, the equation above is not of interest to us. However, we can construct a special case of the problem modeled by the PDE by considering a special case of the physical problem. This special case is the problem of determining the *equilibrium configuration* of the system. This is the configuration in which no part of the bar is in motion. That equilibrium configuration has a displacement function which is purely a function of x, $u(x)$. This causes the t derivative term in the PDE to vanish, and reduces the x derivative to a total derivative

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\, \frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) = f(x)$$

by adding appropriate boundary conditions

$$u(0) = u(l) = 0$$

we have the *BVP form* of the wave equation.

Options for solving the BVP

The differential operator in the BVP form is a symmetric operator that has real eigenvalues and distinct, associated real eigenfunctions. This opens up the option of using the spectral method to solve this BVP. However, unlike the operator we saw earlier in chapter, the eigenfunctions of this operator may not be easy to compute due to the presence of the $k(x)$ term. This puts the spectral method out of reach as a solution method.

For the rest of these notes we will pursue an alternative strategy for solving the BVP. The first step in developing this alternative form is to convert the PDE into an alternative form.

The Energy Form of the Wave Equation

Consider once again the wave equation, this time in the absence of any external forces.

$$\rho(x)\,\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial u(x,t)}{\partial x}\right) = 0$$

(For simplicity here I am going to ignore the factor of A that should appear in both terms. If you like, you can reintroduce that A below to get a more accurate expression.) We saw during the derivation of the wave equation that one can also express the equation as an integral equation. Here we are going to do something similar.

For reasons that will become clear shortly, we start by multiplying both sides of the equation by a factor of $\frac{\partial u}{\partial t}$

$$\rho(x)\,\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = 0$$

We then integrate this over the entire bar.

$$\int_0^l \rho(x)\,\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\,\mathrm{d}x - \int_0^l \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}\,\mathrm{d}x = 0$$

Next, we apply an integration by parts to the second integral to move one of the x derivatives over to the $\frac{\partial u}{\partial t}$ factor.

$$-\int_0^l \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}\,\mathrm{d}x = -\left(k(x)\,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}\Big|_0^l + \int_0^l k(x)\,\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial t\,\partial x}\,\mathrm{d}x$$

Since the ends of the bar never move, $\frac{\partial u(0)}{\partial t}$ is 0 and $\frac{\partial u(l)}{\partial t}$ is 0. Thus

$$-\int_0^l \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}\,\mathrm{d}x = \int_0^l k(x)\,\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial t\,\partial x}\,\mathrm{d}x$$

Next, we apply a trick with derivatives:

$$\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial t\,\partial x} = \frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)$$

This is simply an application of the chain rule. That gives us that

$$-\int_0^l \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}\,\mathrm{d}x = \int_0^l k(x)\,\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial t\,\partial x}\,\mathrm{d}x = \int_0^l k(x)\,\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)\mathrm{d}x$$

We can also do a similar trick with the first of the integrals above.

$$\int_0^l \rho(x)\,\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\,\mathrm{d}x = \int \rho(x)\,\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2\right)\mathrm{d}x$$

Thus we have transformed the original integral form of the wave equation to

$$\int_0^l \rho(x)\,\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2\right)\mathrm{d}x + \int_0^l k(x)\,\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)\mathrm{d}x = 0$$

Noting that the integral is an integral with respect to x, we can factor out the t derivative.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^l \rho(x)\left(\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2\right)\mathrm{d}x+\int_0^l k(x)\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)\mathrm{d}x\right)=0$$

Saying that the derivative of something with respect to t is 0 is equivalent to saying that that thing is a constant over time.

$$\frac{1}{2}\int_0^l \rho(x)\left(\frac{\partial u}{\partial t}\right)^2\mathrm{d}x+\frac{1}{2}\int_0^l k(x)\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x=C$$

Finally, we recognize that both of these integrals have a physical interpretation. The first integral is the expression for the kinetic energy of the bar, while the second computes the potential energy imposed by the stress forces. What we have just derived is an example of the principle of conservation of energy. As the bar moves, energy is constantly being converted between kinetic energy and potential energy, but the energy of the bar as a whole stays constant.

$$\mathit{Kinetic\ Energy}+\mathit{Potential\ Energy}=C$$

The integral equation we developed above is an alternative form of the wave equation, called the *energy form* of the equation.

Potential Energy in the Presense of an External Force

What would the physics look like if we reintroduced an external force? The main effect of the external force is to add a new term to the potential energy. To displace some part of the bar by $u(x,t)$ requires that we do some work to oppose the external force:

$$\mathit{work}=-A\ dx\ u(x,t)\ f(x)$$

By integrating that work over the entire bar we can get a new expression for the potential.

$$E(u)=\frac{1}{2}\int_0^l k(x)\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x-\int_0^l f(x)\ u(x)\,\mathrm{d}x$$

The Equilibrium Configuration is a Configuration of Minimum Potential Energy

Now we consider the equilibrium configuration of the bar in the presence of an external force. As explained above, that equilibrium configuration is the solution to the BVP form of the wave equation.

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)=f(x)$$

$$u(0)=u(l)=0$$

Suppose now we examine some configuration which is not the equilibrium configuration. That is, we add a displacement $v(x)$ to the equilibrium displacement $u(x)$. Here is the expression for the potential energy of that new configuration.

$$E(u+v) = \frac{1}{2}\int_0^l k(x)\left(\frac{du}{dx} + \frac{dv}{dx}\right)^2 dx - \int_0^l f(x)\,(u+v)\,dx$$

How does the new potential energy compare to that of the equilibrium, $E(u)$? We start by expanding some of the terms.

$$\frac{1}{2}\int_0^l k(x)\left(\frac{du}{dx}\right)^2 dx - \int_0^l f(x)\,u\,dx + \int_0^l k(x)\,\frac{du}{dx}\frac{dv}{dx}\,dx - \int_0^l f(x)\,v\,dx + \frac{1}{2}\int_0^l k(x)\left(\frac{dv}{dx}\right)^2 dx$$

The purpose of expanding is to notice that some of these terms form exactly $E(u)$.

$$= E(u) + \int_0^l k(x)\left(\frac{dv}{dx}\right)^2 dx + \left(\int_0^l k(x)\,\frac{du}{dx}\frac{dv}{dx}\,dx - \int_0^l f(x)\,v\,dx\right)$$

What about the other terms here? What can we say about them? We start by working on the first term in parentheses. Integration by parts gives us

$$\int_0^l k(x)\,\frac{du}{dx}\frac{dv}{dx}\,dx = k(x)\,\frac{du}{dx}\,v|_0^l - \int_0^l \frac{d}{dx}\left(k(x)\,\frac{du}{dx}\right) v\,dx = -\int_0^l \frac{d}{dx}\left(k(x)\,\frac{du}{dx}\right) v\,dx$$

The boundary terms vanish because both $u(x)$ and $v(x)$ have to satisfy the boundary conditions that say that displacements must vanish at 0 and at l. Thus we have

$$\int_0^l k(x)\,\frac{du}{dx}\frac{dv}{dx}\,dx - \int_0^l f(x)\,v\,dx = -\int_0^l \left(\frac{d}{dx}\left(k(x)\,\frac{du}{dx}\right) + f(x)\right) v\,dx = 0$$

The last integral vanishes, because the thing being integrated there is the BVP form for the equilibrium configuration. Thus we have that

$$E(u+v) = E(u) + \int_0^l k(x)\left(\frac{dv}{dx}\right)^2 dx > E(u)$$

This tells us that the solution to the BVP has a special property: it is the configuration that minimizes the potential energy of the bar in the presence of an external force.

The Weak Form of the BVP

In the argument above we used the so-called *strong form* of the BVP

$$-\frac{d}{dx}\left(k(x)\,\frac{du}{dx}\right) = f(x)$$

$$u(0) = u(l) = 0$$

to show that

$$\int_0^l k(x)\,\frac{du}{dx}\frac{dv}{dx}\,dx - \int_0^l f(x)\,v\,dx = 0$$

This latter expression is known as the *weak form* of the BVP. This integral equation must hold true for all functions v in $C_D^2[0,l]$. In the discussion that follows in chapter 5 we will be using the weak form as a tool to find approximate solutions to the strong form of the BVP.