Eigenvalues and Eigenvectors

If A is an n by n real-valued matrix we say that λ is an *eigenvalue* of A with associated *eigenvector* $\mathbf{x}$ if

$$A\,\mathbf{x} = \lambda\,\mathbf{x}$$

Typically, a matrix A will have several different eigenvalue, eigenvector pairs. The standard method used to find these pairs is to note that

$$A\,\mathbf{x} = \lambda\,\mathbf{x}$$

if and only if

$$(A - \lambda\, I)\,\mathbf{x} = 0$$

which will happen if and only if the matrix A - λ I is singular. One way to determine whether or not A - λ I is singular is to compute its determinant and check whether or not that determinant is 0. $det(A - \lambda\ I)$ is an n^{th} degree polynomial in λ (called the characteristic polynomial of A), so the problem of finding eigenvalues boils down to the problem of finding roots of that polynomial.

There are a number of technical problems associated with finding eigenvalue, eigenvector pairs.

1. Finding all the roots of a polynomial of high degree may be technically challenging.

2. Not all roots of the characteristic polynomial may be real. Even though all of the entries of A are real, A may still have some complex eigenvalues. Complex eigenvalues will also have complex eigenvectors associated with them.

3. Some roots may be repeated. In the case of repeated roots with multiplicity k we may be able to find k linearly independent eigenvectors for that eigenvalue, or we may be able to find fewer than k associated eigenvectors.

Basis of Eigenvectors

Assuming for the moment that we can find a set of n distinct eigenvectors for some matrix A and also assuming that those eigenvectors form an orthonormal set, we can use those eigenvectors as a basis for the vector space $\mathbb{R}^n$. It turns out that such a basis is especially well suited to help us solve the matrix equation

$$A\,\mathbf{x} = \mathbf{b}$$

The solution method consists of expressing both $\mathbf{x}$ and $\mathbf{b}$ as linear combinations of eigenvectors.

$$A\,(c_1\,\mathbf{u}_1 + c_2\,\mathbf{u}_2 + \cdots + c_n\,\mathbf{u}_n) = d_1\,\mathbf{u}_1 + d_2\,\mathbf{u}_2 + \cdots + d_n\,\mathbf{u}_n$$

Since multiplication by A is a linear operation and the vectors $\mathbf{u}_k$ are all eigenvectors of A we have

$$c_1 \lambda_1 \mathbf{u}_1 + c_2 \lambda_2 \mathbf{u}_2 + \cdots + c_n \lambda_n \mathbf{u}_n = d_1 \mathbf{u}_1 + d_2 \mathbf{u}_2 + \cdots + d_n \mathbf{u}_n$$

Further, since we are assuming that the eigenvectors form a basis and are hence linearly independent, this equation has a solution if and only if

$$c_k \lambda_k = d_k$$

for all k. Since both λ_k and d_k are known, if none of the eigenvalues λ_k are 0 we can solve these equations for all of the c_k and then construct

$$\mathbf{x} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_n \mathbf{u}_n$$

This method, which uses a basis of eigenvectors and their associated eigenvalues to solve a linear system is called the *spectral method*. (The list of eigenvalues of an operator is sometimes referred to as the *spectrum* of that operator, hence *spectral method*.)

The significance of this method is that it applies more broadly to problems involving linear operators. To solve

$$f(\mathbf{x}) = \mathbf{b}$$

for a linear operator f we try to find a basis of eigenvectors and associated eigenvalues:

$$f(\mathbf{u}_k) = \lambda_k \mathbf{u}_k$$

To solve $f(\mathbf{x}) = \mathbf{b}$ we then proceed as above:

$$f(c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_n \mathbf{u}_n) = d_1 \mathbf{u}_1 + d_2 \mathbf{u}_2 + \cdots + d_n \mathbf{u}_n$$

$$c_1 \lambda_1 \mathbf{u}_1 + c_2 \lambda_2 \mathbf{u}_2 + \cdots + c_n \lambda_n \mathbf{u}_n = d_1 \mathbf{u}_1 + d_2 \mathbf{u}_2 + \cdots + d_n \mathbf{u}_n$$

This can be solved as above by solving the equations

$$c_k \lambda_k = d_k$$

for the unknowns c_k.

An Important Special Case

There is one very important special class of matrices for which the program outlined above works very nicely. These are the n by n *symmetric*, real-valued matrices. A matrix A is symmetric if it is equal to its own transpose. The key theorem that tells us that real-valued symmetric matrices are nice is the following.

Theorem If A is a real-valued, symmetric, n by n matrix, A has a complete set of associated real-valued eigenvectors. Further, eigenvectors corresponding to distinct eigenvalues are orthogonal to each other. Thus, it is possible to construct an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.