Using the free-space Laplacian Green's function to solve a bounded problem

In section 11.5 we derived Green's functions to solve the free-space Poisson equation

$$-\Delta\, u(\mathbf{x}) = f(\mathbf{x})$$

Our goal in section 11.6 is to solve the Poisson equation in bounded regions. The simplest form of this problem is the Poisson equation on some bounded region Ω with Dirichlet boundary conditions on $\delta\Omega$:

$$-\Delta\, u(\mathbf{x}) = f(\mathbf{x})$$

$$u(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \delta\Omega$$

The free-space Green's function allows us to construct a function

$$u_1(x) = \int_\Omega G(x;y)\, f(y)\, d\,y$$

This function satisfies the differential equation

$$-\Delta\, u_1(\mathbf{x}) = f(\mathbf{x})$$

but there is no reason to expect that it also satisfies the boundary condition.

The question now is whether or not we can modify $u_1(\mathbf{x})$ to produce a solution that satisfies the boundary condition and the differential equation.

The obvious guess is to try something like this - find a function $v_\Omega(\mathbf{x};\mathbf{y})$ that satisfies

$$-\Delta\, v_\Omega(\mathbf{x};\mathbf{y}) = 0$$

$$v_\Omega(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}) \text{ for } \mathbf{x} \in \delta\Omega$$

and then construct a new Green's function

$$G_\Omega(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}) - v_\Omega(\mathbf{x};\mathbf{y})$$

and then show that

$$u_2(\mathbf{x}) = \int_\Omega G_\Omega(\mathbf{x};\mathbf{y})\, f(\mathbf{y})\, d\mathbf{y}$$

satisfies

$$-\Delta\, u_2(\mathbf{x}) = f(\mathbf{x})$$

$$u_2(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \delta\Omega$$

Here now is the proof that this produces the desired result. First, we confirm that $u_2(\mathbf{x})$ satisfies

$$-\Delta\ u_2(\mathbf{x}) = f(\mathbf{x})$$

We have by linearity that

$$-\Delta\ u_2(\mathbf{x}) = -\Delta\ (\int_\Omega G_\Omega(\mathbf{x};\mathbf{y})\ f(\mathbf{y})\ d\mathbf{y}) = -\Delta(\int_\Omega (G(\mathbf{x};\mathbf{y}) - v_\Omega(\mathbf{x};\mathbf{y}))\ f(\mathbf{y})\ d\mathbf{y})$$

$$= -\Delta(\int_\Omega G(\mathbf{x};\mathbf{y})\ f(\mathbf{y})\ d\mathbf{y}) - \Delta(\int_\Omega v_\Omega(\mathbf{x};\mathbf{y})\ f(\mathbf{y})\ d\mathbf{y})$$

$$= -\Delta\ u_1(\mathbf{x}) - \Delta(\int_\Omega v_\Omega(\mathbf{x};\mathbf{y})\ f(\mathbf{y})\ d\mathbf{y})$$

$$= f(\mathbf{x}) - \Delta(\int_\Omega v_\Omega(\mathbf{x};\mathbf{y})\ f(\mathbf{y})\ d\mathbf{y})$$

Note that in this final integral we are integrating with respect to $\mathbf{y}$ but taking the derivative with respect to $\mathbf{x}$. This allows us to move the Laplacian inside the integral to obtain

$$-\ \Delta(\int_\Omega v_\Omega(\mathbf{x};\mathbf{y})\ f(\mathbf{y})\ d\mathbf{y}) = \int_\Omega (-\ \Delta v_\Omega(\mathbf{x};\mathbf{y}))\ f(\mathbf{y})\ d\mathbf{y} = \int_\Omega (0)\ f(\mathbf{y})\ d\mathbf{y} = 0$$

Thus we see that $u_2(\mathbf{x})$ satisfies

$$-\Delta\ u_2(\mathbf{x}) = f(\mathbf{x})$$

The final thing to check is that $u_2(\mathbf{x})$ satisfies the boundary condition. Note that by definition

$$v_\Omega(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y})$$

for all $\mathbf{x}$ on the boundary, so that

$$u_2(\mathbf{x}) = \int_\Omega G_\Omega(\mathbf{x};\mathbf{y})\ f(\mathbf{y})\ d\mathbf{y} = \int_\Omega (G(\mathbf{x};\mathbf{y}) - v_\Omega(\mathbf{x};\mathbf{y}))\ f(\mathbf{y})\ d\mathbf{y} = \int_\Omega (G(\mathbf{x};\mathbf{y}) - G(\mathbf{x};\mathbf{y}))\ f(\mathbf{y})\ d\mathbf{y} = 0$$

for all $\mathbf{x}$ on the boundary.

More difficult boundary conditions

In the textbook the author works through the details of the argument for the more general boundary value problem

$$-\Delta\ u(\mathbf{x}) = f(\mathbf{x})$$

$$\alpha_1\ u(\mathbf{x}) + \alpha_2\ \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} = 0 \text{ for } \mathbf{x} \in \delta\Omega$$

The strategy is similar for this case. Once again, we use the free-space Green's function to construct a second problem

$$-\Delta\ v_\Omega(\mathbf{x}) = f(\mathbf{x})$$

$$\alpha_1\ v_\Omega(\mathbf{x}) + \alpha_2\ \frac{\partial v_\Omega(\mathbf{x})}{\partial \mathbf{n}} = G(\mathbf{x};\mathbf{y}) \text{ for } \mathbf{x} \in \delta\Omega$$

and then construct a new Green's function

$$G_\Omega(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}) \text{ - } v_\Omega(\mathbf{x};\mathbf{y})$$

and then show that

$$u_2(\mathbf{x}) = \int_\Omega G_\Omega(\mathbf{x};\mathbf{y})\ f(\mathbf{y})\ d\mathbf{y}$$

satisfies

$$\text{-}\Delta\ u_2(\mathbf{x}) = f(\mathbf{x})$$

$$\alpha_1\ u(\mathbf{x}) + \alpha_2\ \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} = 0 \text{ for } \mathbf{x} \in \delta\Omega$$

The proof for this case is more involved, and I refer you to the textbook for the details.

One remaining problem

We have now reduced the problem of constructing Green's functions for bounded domains to solving problems of the form

$$\text{-}\Delta\ v_\Omega(\mathbf{x};\mathbf{y}) = 0$$

$$v_\Omega(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}) \text{ for } \mathbf{x} \in \delta\Omega$$

where $G(\mathbf{x};\mathbf{y})$ is the free-space Green's function and $\mathbf{y}$ is a point in Ω. The problem that remains here is that these problems are not necessarily trivial to solve. We find once again here that we can only really solve these problems effectively in cases where the geometry of Ω allows us to use a simple solution method such as separation of variables. Another alternative is to try to solve more directly for $v_\Omega(\mathbf{x};\mathbf{y})$ by applying an argument that leverages the special geometry of the region. The next section will demonstrate one such alternative method.

Solving boundary conditions by the method of images

We now restrict our consideration to a version of the problem above with simple geometry. Specifically, we will assume that the region Ω is a circle of radius R centered at the origin.

The situation now is that we are looking for a function $v_\Omega(\mathbf{x};\mathbf{y})$ that satisfies

$$\text{-}\Delta\ v_\Omega(\mathbf{x};\mathbf{y}) = 0$$

$$v_\Omega(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}) \text{ for } \mathbf{x} \in \delta\Omega$$

As a first guess, we might try doing

$$v_1(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y})$$

this certainly satisfies

$$v_1(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}) \text{ for } \mathbf{x} \in \delta\Omega$$

but doesn't quite solve the PDE, because

$$-\Delta\, v_1(\mathbf{x};\mathbf{y}) = -\Delta\, G(\mathbf{x};\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \neq 0$$

For our second attempt, we try to work around the latter problem by instead trying

$$v_2(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}^*)$$

where $G(\mathbf{x};\mathbf{y})$ is the usual free-space Green's function and $\mathbf{y}^*$ is an *image point* chosen to lie outside the region Ω. This solves the PDE problem, because

$$-\Delta\, v_2(\mathbf{x};\mathbf{y}) = -\Delta\, G(\mathbf{x};\mathbf{y}^*) = \delta(\mathbf{x} - \mathbf{y}^*) = 0 \text{ for all } \mathbf{x} \in \Omega$$

However, we have no guarantee that v_2 also satisfies the necessary boundary condition:

$$v_2(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}^*) = G(\mathbf{x};\mathbf{y}) \text{ for } \mathbf{x} \in \delta\Omega$$

In fact, it turns out that there is no way to choose $\mathbf{y}^*$ outside the circle so that $G(\mathbf{x};\mathbf{y}^*) = G(\mathbf{x};\mathbf{y})$ everywhere around the perimeter of the circle.

What we have to do instead is allow for an additional 'fudge factor'. Specifically, we assume for our third attempt that the function that works is

$$v_3(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}^*) + c(\mathbf{y})$$

The additional term is the required 'fudge factor' needed to get everything to work.

Once again, we check that

$$-\Delta\, v_3(\mathbf{x};\mathbf{y}) = -\Delta\, G(\mathbf{x};\mathbf{y}^*) - \Delta c(\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}^*) - 0 = 0 \text{ for all } \mathbf{x} \in \Omega$$

and now require that

$$v_3(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}^*) + c(\mathbf{y}) = G(\mathbf{x};\mathbf{y}) \text{ for } \mathbf{x} \in \delta\Omega$$

This final form can be made to work. Here are the details.

We assume that we are working in two dimensions and that Ω is a circle of radius R centered at the origin. The appropriate free-space Green's function is

$$G(\mathbf{x};\mathbf{y}) = -\frac{1}{2\,\pi} \ln(||\mathbf{x}\text{-}\mathbf{y}||)$$

We seek a $\mathbf{y}^*$ outside the circle so that

$$-\frac{1}{2\,\pi}\ \ln(||\mathbf{x}\text{-}\mathbf{y}^*||) + c(\mathbf{y}) = G(\mathbf{x};\mathbf{y}^*) + c(\mathbf{y}) \ = G(\mathbf{x};\mathbf{y}) = -\frac{1}{2\,\pi}\ \ln(||\mathbf{x}\text{-}\mathbf{y}||)$$

This is equivalent to

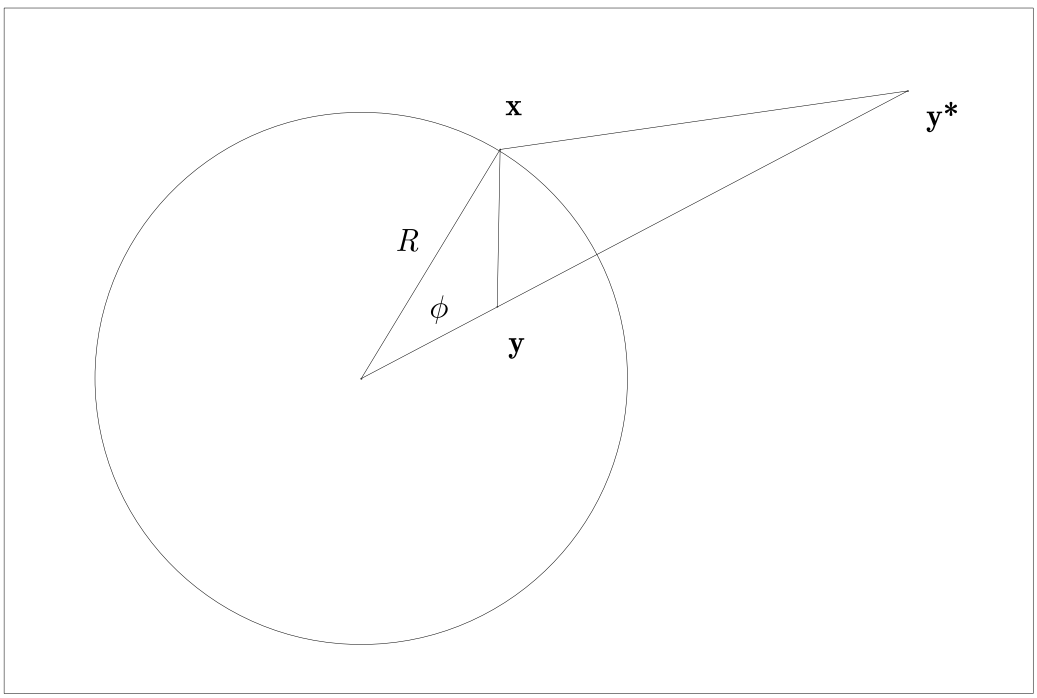
$$c(\mathbf{y}) = -\frac{1}{2\,\pi}\ \ln\left(\frac{||\mathbf{x}\text{-}\mathbf{y}||}{||\mathbf{x}\text{-}\mathbf{y}^*||}\right)$$

The key requirement here is that the expression on the right hand side be independent of **x** for all **x** on the perimeter of the circle. This can be achieved if we demand that

$$\frac{||\mathbf{x}\text{-}\mathbf{y}||}{||\mathbf{x}\text{-}\mathbf{y}^*||} = m(\mathbf{y}) \text{ for } \mathbf{x} \in \delta\Omega$$

A final helping hint comes from a symmetry argument. We demand that when **y** and **y*** are expressed in polar coordinates they both have the same polar angle θ. That is, both **y** and **y*** lie on the same ray running from the origin to **y**.

Here is a picture illustrating the situation.



Consider the triangle formed by the origin, **x**, and **y** and the triangle formed by the origin, **x**, and **y***. Both of these triangles share the angle labeled by ϕ in the picture. In both cases we can write down expressions involving $||\mathbf{x}\text{-}\mathbf{y}||$ and $||\mathbf{x}\text{-}\mathbf{y}^*||$ by using the law of cosines.

$$R^2 + {r_y}^2 - 2\ R\ r_y \cos\phi = ||\mathbf{x}\text{-}\mathbf{y}||^2$$

$$R^2 + {r_{y^*}}^2 - 2\ R\ r_{y^*} \cos\phi = ||\mathbf{x}\text{-}\mathbf{y}^*||^2$$

The earlier condition that

$$\frac{||\mathbf{x}\text{-}\mathbf{y}||}{||\mathbf{x}\text{-}\mathbf{y*}||} = m(\mathbf{y})$$

now leads to a requirement that

$$\frac{R^2 + {r_y}^2 \text{ - } 2\ R\ r_y \cos\phi}{R^2 + {r_{y*}}^2 \text{ - } 2\ R\ r_{y*} \cos\phi} = (m(\mathbf{y}))^2$$

for all angles ϕ. In particular, it must be true for $\phi = 0$ and $\phi = \pi$.

$$\frac{R^2 + {r_y}^2 \text{ - } 2\ R\ r_y}{R^2 + {r_{y*}}^2 \text{ - } 2\ R\ r_{y*}} = (m(\mathbf{y}))^2$$

$$\frac{R^2 + {r_y}^2 + 2\ R\ r_y}{R^2 + {r_{y*}}^2 + 2\ R\ r_{y*}} = (m(\mathbf{y}))^2$$

These two equations can be rewritten

$$R^2 + {r_y}^2 \text{ - } 2\ R\ r_y = (m(\mathbf{y}))^2\ (R^2 + {r_{y*}}^2 \text{ - } 2\ R\ r_{y*})$$

$$R^2 + {r_y}^2 + 2\ R\ r_y = (m(\mathbf{y}))^2\ (R^2 + {r_{y*}}^2 + 2\ R\ r_{y*})$$

Subtracting these two equations gives

$$4\ R\ r_y = (m(\mathbf{y}))^2\ (4\ R\ r_{y*})$$

or

$$\frac{r_y}{r_{y*}} = (m(\mathbf{y}))^2$$

Another relevant equation comes from setting $\phi = \pi/2$.

$$\frac{R^2 + {r_y}^2}{R^2 + {r_{y*}}^2} = (m(\mathbf{y}))^2 = \frac{r_y}{r_{y*}}$$

Solving this equation for r_{y*} gives

$$r_{y*} = \frac{R^2}{r_y}$$

and subsequently

$$m(\mathbf{y}) = \frac{R}{r_y}$$

Since $r_y = ||\mathbf{y}||$ we have that

$$\mathbf{y*} = \frac{R^2}{||\mathbf{y}||^2}\,\mathbf{y}$$

Putting this all together we have that

$$v_3(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y*}) + c(\mathbf{y}) = -\frac{1}{2\,\pi}\ln(||\mathbf{x}\text{-}\mathbf{y*}||) - \frac{1}{2\,\pi}\ln\,(m(\mathbf{y}))$$

$$= -\frac{1}{2\,\pi}\ln(||\mathbf{x}\text{-}\frac{R^2}{||\mathbf{y}||^2}\,\mathbf{y}||) - \frac{1}{2\,\pi}\ln\left(\frac{R}{||\mathbf{y}||}\right)$$

Now that we have computed $v_3(\mathbf{x};\mathbf{y})$ we can compute the desired Green's function by doing

$$G_\Omega(\mathbf{x};\mathbf{y}) = G(\mathbf{x};\mathbf{y}) - v_3(\mathbf{x};\mathbf{y})$$

$$= -\frac{1}{2\,\pi}\ln(||\mathbf{x}\text{-}\mathbf{y}||) + \frac{1}{2\,\pi}\ln(||\mathbf{x}\text{-}\frac{R^2}{||\mathbf{y}||^2}\,\mathbf{y}||) + \frac{1}{2\,\pi}\ln\left(\frac{R}{||\mathbf{y}||}\right)$$

$$= \frac{1}{2\,\pi}\ln\left(\frac{||\,||\mathbf{y}||^2\mathbf{x}\text{-}\,R^2\,\mathbf{y}||}{R\,||\mathbf{y}||\;||\mathbf{x}\text{-}\mathbf{y}||}\right)$$