Free-space Green's function for the Laplacian

One way to construct a Green's function for the Laplacian in multiple spatial dimensions is to solve the PDE

$$-\Delta\ u = \delta(\mathbf{x}\text{-}\mathbf{y})$$

with appropriate boundary conditions.

The solution to this equation

$$u(\mathbf{x}) = G(\mathbf{x};\mathbf{y})$$

is the Green's function for the Laplacian.

Depending on the region Ω and the boundary conditions we impose we will of course get different Green's functions. The simplest scenario is the so-called free-space scenario in which we seek to solve the PDE on an unbounded spatial region. In that scenario there are no boundary conditions per-se, but we may want to impose an auxiliary condition on the solution, such as requiring that the solution be bounded at infinity.

Translational and Rotational Invariance

One key to solving the PDE above simply is to note that the Laplace operator is translation invariant. That is, if

$$-\Delta = -\frac{\partial^2}{\partial x_1{}^2} - \frac{\partial^2}{\partial x_2{}^2}$$

in the usual coordinate system and we introduce a change of coordinates

$$z_1 = x_1 + y_1$$

$$z_2 = x_2 + y_2$$

where y_1 and y_2 are constants, then in that new coordinate system we have that

$$-\Delta = -\frac{\partial^2}{\partial z_1{}^2} - \frac{\partial^2}{\partial z_2{}^2}$$

An important consequence of this property is that if $\psi(\mathbf{x})$ is a solution of

$$-\Delta\ u = f(\mathbf{x})$$

then $\psi(\mathbf{x}\text{-}\mathbf{y})$ is a solution of

$$-\Delta\ u = f(\mathbf{x}\text{-}\mathbf{y})$$

This observation is immediately useful to us, because now instead of worrying about how to solve

$$-\Delta\ u = \delta(\mathbf{x}\text{-}\mathbf{y})$$

we can instead work on solving

$$-\Delta\ u = \delta(\mathbf{x})$$

and then replace the solution $u(\mathbf{x})$ with $u(\mathbf{x}\text{-}\mathbf{y})$ to get the desired $G(\mathbf{x};\mathbf{y})$.

A second observation that will be useful to us is that if the forcing function in the free-space Poisson equation

$$-\Delta\ u = f(\mathbf{x})$$

is radially symmetric then we can guess that the solution is likewise radially symmetric:

$$u(\mathbf{x}) = \phi(r)$$

Using the Laplacian expressed in polar coordinates, this gives that

$$-\Delta\ u = -\frac{\partial^2\phi(r)}{\partial r^2} - \frac{1}{r}\frac{\partial\phi(r)}{\partial r} - \frac{1}{r^2}\frac{\partial^2\phi(r)}{\partial\theta^2} = f(r)$$

or

$$-\frac{\mathrm{d}^2\phi(r)}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}\phi(r)}{\mathrm{d}r} = f(r)$$

This conveniently reduces the PDE to an ODE. In our case, we need to solve

$$-\frac{\mathrm{d}^2 g(r)}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}g(r)}{\mathrm{d}r} = \delta(r)$$

Solving for g(r)

One approach to solving this ODE is to try instead to solve

$$-\frac{\mathrm{d}^2 g(r)}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}g(r)}{\mathrm{d}r} = 0$$

and see if the solution to that problem just happens to also solve the ODE with the delta function on the right hand side. One solution to this ODE is

$$g(r) = c_1 \ln(r)$$

Given this, and given the considerations above based on tranlational and rotational symmetry of the Laplace operator, we would like to advance an educated guess that the free-space Green's function takes the form

$$G(\mathbf{x};\mathbf{y}) = c_1 \ln(||\mathbf{x}\text{-}\mathbf{y}||)$$

Verifying the Green's function

Because the original problem we set out to solve

$$-\Delta\ u = \delta(\mathbf{x}\text{-}\mathbf{y})$$

contains a delta function, we can not verify directly that the proposed solution is a solution by simply substituting into the equation and simplifying. Instead, we will seek to show that the proposed solution satisfies the weak form of the PDE.

To construct the weak form, we select an arbitrary test function $v(x)$ and construct

$$\int_\Omega (-\Delta u(\mathbf{x}))\, v(\mathbf{x})\ d\mathbf{x} = \int_\Omega \delta(\mathbf{x}\text{-}\mathbf{y})\, v(\mathbf{x})\ d\mathbf{x} = v(\mathbf{y})$$

On small technical concern here is that we are working on an unbounded region in space. To eliminate concerns with the convergence of the integral on the left, we might want to start by using test functions $v(\mathbf{x})$ with bounded support.

The next step is to apply Green's first identity to the integral on the left.

$$\int_\Omega (-\Delta u(\mathbf{x}))\, v(\mathbf{x})\ d\mathbf{x} = -\int_{\delta\Omega} \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}}\ v(\mathbf{x})\ d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})\ d\mathbf{x}$$

Because we are operating on a free space region and we have also assumed that the test function v(x) has bounded support, the first integral on the right vanishes. Thus we have the weak form of the PDE

$$\int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})\ d\mathbf{x} = v(\mathbf{y})$$

We now attempt to verify that $u(\mathbf{x}) = c_1 \ln(||\mathbf{x}\text{-}\mathbf{y}||)$ satisfies this weak form equation. One remaining technical problem is that

$$\nabla u(\mathbf{x}) = -c_1\ \frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x} - \mathbf{y}||^2}$$

is singular at $\mathbf{x} = \mathbf{y}$. We will work around this problem by introducing a region

$$\Omega_\epsilon = \{x \in \mathbb{R}^2 \mid \epsilon < ||\mathbf{x} - \mathbf{y}|| < R\}$$

and demonstrating first that

$$\int_{\Omega_\epsilon} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})\ d\mathbf{x} = -\int_{\Omega_\epsilon} c_1\ \frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x}\text{-}\mathbf{y}||^2} \cdot \nabla v(\mathbf{x})\ d\mathbf{x} = v(\mathbf{y})$$

If we can show that this is true for all ϵ and all R, we can then take the limit as $\epsilon \to 0$ and $R \to \infty$

to obtain the result we want.

The first step in achieving our result is to apply Green's identity to the integral.

$$-\int_{\Omega_\epsilon} c_1 \frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x}\text{-}\mathbf{y}||^2} \cdot \nabla v(\mathbf{x})\, d\mathbf{x} = -\int_{\delta\Omega_e} v(\mathbf{x})\, c_1 \frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x}\text{-}\mathbf{y}||^2} \cdot \mathbf{n}\ \mathbf{dx} + \int_\Omega \nabla \cdot \left(c_1 \frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x}\text{-}\mathbf{y}||^2} \right) v(\mathbf{x})\, d\mathbf{x}$$

The second term on the right vanishes because

$$\nabla \cdot \left(c_1 \frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x}\text{-}\mathbf{y}||^2} \right) = 0$$

This leaves us with only the boundary integral to worry about. The boundary in this case consists of two circles, $||\mathbf{x}\text{-}\mathbf{y}|| = \epsilon$ and $||\mathbf{x}\text{-}\mathbf{y}|| = R$. We can deal with the outer boundary by noting that the test function $v(\mathbf{x})$ has compact support, so that if we make R large enough $v(\mathbf{x})$ will vanish on that outer ring.

On the inner ring, we have that

$$\mathbf{n} = -\frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x} - \mathbf{y}\,||}$$

so that

$$\frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x}\text{-}\mathbf{y}||^2} \cdot \mathbf{n} = -\frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x}\text{-}\mathbf{y}||^2} \cdot \frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x} - \mathbf{y}\,||} = -\frac{1}{||\mathbf{x} - \mathbf{y}\,||} = -\frac{1}{\epsilon}$$

We now have that

$$-\int_{\Omega_\epsilon} c_1 \frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x} - \mathbf{y}||^2} \cdot \nabla v(\mathbf{x})\, d\mathbf{x} = \frac{c_1}{\epsilon} \int_{S_e} v(\mathbf{x})\, d\mathbf{x}$$

For $\mathbf{x}$ close to $\mathbf{y}$, we have that $v(\mathbf{x}) \approx v(\mathbf{y})$ and the integral over S_ϵ devolves to a simple integral of $v(\mathbf{y})$ over the circle of radius ϵ centered at $\mathbf{y}$.

$$\frac{c_1}{\epsilon} \int_{S_e} v(\mathbf{x})\, d\mathbf{x} \approx \frac{c_1}{\epsilon}\, 2\, \pi\, \epsilon\, v(\mathbf{y}) = c_1\, 2\, \pi\, v(\mathbf{y})$$

This is only approximate equality, but in the limit as $\epsilon \to 0$ we will have exact equality. Since our goal was to get

$$v(\mathbf{y}) = \int_\Omega (-\Delta u(\mathbf{x}))\, v(\mathbf{x})\, d\mathbf{x} = -\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} c_1 \frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x} - \mathbf{y}||^2} \cdot \nabla v(\mathbf{x})\, d\mathbf{x} = c_1\, 2\, \pi\, v(\mathbf{y})$$

We see that this will work if we simply select

$$c_1 = \frac{1}{2\, \pi}$$

We have now established that the Green's function for the free-space Laplacian in two dimensions is

$$G(\mathbf{x};\mathbf{y}) = \frac{1}{2\pi}\ln(||\mathbf{x}\text{-}\mathbf{y}||)$$

Green's Function for the Laplacian in Three Dimensions

The argument for three spatial dimensions is similar, except for one key step. A key step in the argument above for two dimensions was to note that since the delta function is a radially symmetric function centered at $x = y$ we would do well to do a change of variables to move the origin to y and then convert to polar coordinates. The argument in three dimensions is similar, except that the form of the Laplacian in spherical coordinates is a little different. Instead of

$$-\Delta\, u = -\frac{\partial^2 u(r,\theta)}{\partial r^2} - \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2}$$

we have

$$-\Delta\, u = -\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u(\rho,\phi,\theta)}{\partial \rho}\right) - \frac{1}{\rho^2 \sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u(\rho,\phi,\theta)}{\partial \phi}\right) - \frac{1}{\rho^2\sin^2(\phi)}\frac{\partial^2 u(\rho,\phi,\theta)}{\partial \theta^2}$$

Once again we use the argument that the solution must be radially symmetric, i. e., $u(\rho,\phi,\theta) = \psi(\rho)$, which reduces the problem to an ODE

$$-\frac{1}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}\psi(\rho)}{\mathrm{d}\rho}\right) = 0,\ \rho > 0$$

This ODE has solution

$$g(\rho) = \frac{c_1}{\rho} + c_2$$

By an argument similar to the argument used above, we can show that $c_1 = 1/4\pi$ and $c_2 = 0$. Thus, the free-space Green's function in three dimensions is

$$G(\mathbf{x};\mathbf{y}) = \frac{1}{4\pi\, ||\mathbf{x} - \mathbf{y}||}$$