Solving the Poisson Equation by the Finite Element Method

We wish to solve the Poisson equation

$$-\Delta\ u = f(x,y)$$

on a region in space Ω that does not have a simple enough geometry to allow solution by separation of variables. For simplicity we will assume that the solution obeys Dirichlet boundary conditions on the boundary $\delta\Omega$.

We will follow the familiar Galerkin method, which begins with the construction of the weak form of the equation:

$$\int_\Omega(-\Delta u(\mathbf{x}))\ v(\mathbf{x})\ d\mathbf{x} = \int_\Omega f(\mathbf{x})\ v(\mathbf{x})\ d\mathbf{x}$$

On the left we use Green's first identity.

$$\int_\Omega(-\Delta u(\mathbf{x}))\ v(\mathbf{x})\ d\mathbf{x} = -\int_{\delta\Omega}\frac{\partial u}{\partial n}(\mathbf{x})\ v(\mathbf{x})\ d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x})\ \cdot\ \nabla v(\mathbf{x})\ d\mathbf{x}$$

Because $v(\mathbf{x})$ is a test function that satisfies the Dirichlet boundary conditions, the first integral vanishes.

$$\int_\Omega \nabla u(\mathbf{x})\ \cdot\ \nabla v(\mathbf{x})\ d\mathbf{x} = \int_\Omega f(\mathbf{x})\ v(\mathbf{x})\ d\mathbf{x}$$

As before, we introduce a bilinear form

$$a(u,v) = \int_\Omega \nabla u(\mathbf{x})\ \cdot\ \nabla v(\mathbf{x})\ d\mathbf{x}$$

and recast the problem as having to find the $u(\mathbf{x})$ that satisfies

$$a(u,v) = (f,v)$$

for all test functions $v(\mathbf{x})$.

Next, we restrict our space of test functions to a finite subspace spanned by a set of functions $\phi_i(\mathbf{x})$ and try to construct the function

$$u_N(\mathbf{x}) = \sum_{n=1}^{N} u_i\ \phi_i(\mathbf{x})$$

that satisfies

$$a(u_N,\phi_i) = (f,\phi_i)$$

for all i. Once again, this leads to an N by N system of equations

$$K\,\mathbf{u} = \mathbf{f}$$

where

$$K_{i,j} = a(\phi_i,\phi_j)$$

and

$$\mathbf{f}_i = (f\,,\,\phi_i)$$

As before, it is to our advantage to select a set of basis functions ϕ_i that have the effect of making the K matrix as simple as possible. Once again, we will seek to construct a family of "spike" functions that each have limited support in Ω and yet collectively the supports of all the ϕ_i functions covers all of Ω.

The construction of appropriate spike functions in $\mathbb{R}^n$ is somewhat more challenging technically. In the accompanying Mathematica notebook I will show how to construct such a set of spike functions to cover a region in $\mathbb{R}^2$.