Eigenfunctions of the Laplace Operator on a Rectangular Domain

Consider the Laplace operator

$$L_D\, u = -\,\Delta\, u$$

defined on a space of functions satisfying Dirichlet boundary conditions on a rectangular region $\Omega = \{x,y|\ 0 \le x \le l_1\ ,\ 0 \le y \le l_2\}$ in $\mathbb{R}^2$:

$$u \in \{\ v \in C^2[\Omega]\ |\ v(0,y) = v(l_1,y) = 0,\ v(x,0) = v(x,l_2) = 0\ \}$$

To begin the process of computing eigenfunctions and eigenvalues of this operator we take a cue from the geometry of the region Ω and assume that eigenfunctions take the form

$$u(x,y) = u_1(x)\ u_2(y)$$

This leads to a solution method known as *separation of variables.*

Substituting this assumption into the eigenvalue equation

$$-\,\Delta\, u = \lambda\, u$$

leads to

$$-\frac{\partial^2 u_1(x)\ u_2(y)}{\partial x^2} - \frac{\partial^2 u_1(x)\ u_2(y)}{\partial y^2} = \lambda\ u_1(x)\ u_2(y)$$

or

$$-\,u_2(y)\ \frac{\mathrm{d}^2 u_1(x)}{\mathrm{d}x^2} - u_1(x)\ \frac{\mathrm{d}^2 u_2(y)}{\mathrm{d}y^2} = \lambda\ u_1(x)\ u_2(y)$$

or

$$-\,\frac{1}{u_1(x)}\ \frac{\mathrm{d}^2 u_1(x)}{\mathrm{d}x^2} - \frac{1}{u_2(y)}\ \frac{\mathrm{d}^2 u_2(y)}{\mathrm{d}y^2} = \lambda$$

Since the first term on the left is independent of y and the second term is independent of x, the only way for these two terms to always sum to a constant is for each term separately to equal a constant. We introduce constants θ_1 and θ_2 and demand that

$$\theta_1 + \theta_2 = \lambda$$

$$-\,\frac{1}{u_1(x)}\ \frac{\mathrm{d}^2 u_1(x)}{\mathrm{d}x^2} = \theta_1$$

$$-\frac{1}{u_2(y)}\frac{d^2 u_2(y)}{dy^2} = \theta_2$$

This separates the original PDE into a pair of ODEs. The original Dirichlet conditions

$$u(0,y) = u(l_1,y) = 0,\ u(x,0) = u(x,l_2) = 0$$

also separate cleanly to produce Dirichlet conditions for each of the ODEs:

$$u_1(0) = u_1(l_1) = 0$$

$$u_2(0) = u_2(l_2) = 0$$

The first ODE

$$\frac{d^2 u_1(x)}{dx^2} + \theta_1\, u_1(x) = 0$$

$$u_1(0) = u_1(l_1) = 0$$

has solutions

$$u_1(x) = \sin\left(\frac{n\,\pi\, x}{l_1}\right),\ n = 1,2,...$$

which leads to

$$\theta_1 = \frac{n^2\,\pi^2}{l_1^2}$$

The second ODE has solutions

$$u_2(y) = \sin\left(\frac{m\,\pi\, y}{l_2}\right),\ m = 1,2,...$$

which leads to

$$\theta_2 = \frac{m^2\,\pi^2}{l_2^2}$$

Thus we see that the eigenfunctions of the original PDE are

$$u(x) = \sin\left(\frac{n\,\pi\, x}{l_1}\right)\sin\left(\frac{m\,\pi\, y}{l_2}\right)$$

with associated eigenvalues

$$\lambda = \theta_1 + \theta_2 = \frac{n^2 \pi^2}{l_1^2} + \frac{m^2 \pi^2}{l_2^2}$$

Solving the Poisson Equation with Dirichlet Boundary Conditions

As an application of the ideas we developed above we now solve an instance of the Poisson equation with Dirichlet boundary conditions on $\Omega = \{x,y|\ 0 \le x \le l_1\ ,\ 0 \le y \le l_2\}$:

$$-\Delta\, u = f(x,y)$$

As usual, we start by assuming that the solution is a sum over eigenfunctions. Since the eigenfunctions are indexed by two integer indices, we have to sum over all possible values of these two indices:

$$u(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{n,m} \sin\left(\frac{n\,\pi}{l_1}\,x\right) \sin\left(\frac{m\,\pi}{l_2}\,y\right)$$

We likewise assume that the forcing function can be written as a sum of eigenfunctions

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{n,m} \sin\left(\frac{n\,\pi}{l_1}\,x\right) \sin\left(\frac{m\,\pi}{l_2}\,y\right)$$

where the double Fourier sine coefficients of $f(x,y)$ can be computed via

$$c_{n,m} = \frac{2}{l_2}\,\frac{2}{l_1} \int_0^{l_2} \int_0^{l_1} f(x,y) \sin\left(\frac{n\,\pi}{l_1}\,x\right) \sin\left(\frac{m\,\pi}{l_2}\,y\right)\, dx\, dy$$

(It is interesting to note here that this method works even in cases in which the forcing function $f(x,y)$ itself can not be written as the product of a function that depends only on x and a function that depends only on y.)

Substituting all of this into the PDE gives

$$-\Delta\, u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{n^2 \pi^2}{l_1^2} + \frac{m^2 \pi^2}{l_2^2}\right) a_{n,m} \sin\left(\frac{n\,\pi}{l_1}\,x\right) \sin\left(\frac{m\,\pi}{l_2}\,y\right)$$

$$= f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{n,m} \sin\left(\frac{n\,\pi}{l_1}\,x\right) \sin\left(\frac{m\,\pi}{l_2}\,y\right)$$

From this we read off that

$$a_{n,m} = \frac{c_{n,m}}{\dfrac{n^2 \pi^2}{l_1^2} + \dfrac{m^2 \pi^2}{l_2^2}}$$

A time dependent problem

We now consider an example involving the wave equation on a rectangular region $\Omega = \{x,y|\ 0 \le x \le l_1\ ,\ 0 \le y \le l_2\}$

$$\frac{\partial^2 u}{\partial t^2} - c^2\ \Delta u = 0$$

$$u(x,y,0) = \psi(x,y)$$

$$\frac{\partial u}{\partial t}(x,y,0) = 0$$

$$u(0,y,t) = u(l_1,y,t) = 0,\ u(x,0,t) = u(x,l_2,t) = 0$$

We assume that solutions take the form

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m}(t)\ \sin\left(\frac{n\ \pi}{l_1}\ x\right)\sin\left(\frac{m\ \pi}{l_2}\ y\right)$$

and that the initial function can be written as a combination of eigenfunctions

$$\psi(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{n,m}\ \sin\left(\frac{n\ \pi}{l_1}\ x\right)\sin\left(\frac{m\ \pi}{l_2}\ y\right)$$

where

$$c_{n,m} = \frac{2}{l_2}\ \frac{2}{l_1}\ \int_0^{l_2}\int_0^{l_1}\psi(x,y)\sin\left(\frac{n\ \pi}{l_1}\ x\right)\sin\left(\frac{m\ \pi}{l_2}\ y\right)\,d\,x\,d\,y$$

The initial condition leads immediately to

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m}(0)\ \sin\left(\frac{n\ \pi}{l_1}\ x\right)\sin\left(\frac{m\ \pi}{l_2}\ y\right) = \psi(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{n,m}\ \sin\left(\frac{n\ \pi}{l_1}\ x\right)\sin\left(\frac{m\ \pi}{l_2}\ y\right)$$

$$\frac{\partial u}{\partial t}(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mathrm{d}a_{n,m}}{\mathrm{d}t}(0)\ \sin\left(\frac{n\ \pi}{l_1}\ x\right)\sin\left(\frac{m\ \pi}{l_2}\ y\right) = 0$$

or

$$a_{n,m}(0) = c_{n,m}$$

$$\frac{\mathrm{d}a_{n,m}}{\mathrm{d}t}(0) = 0$$

Substituting the expression for $u(x,y,t)$ into the PDE gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mathrm{d}^2 a_{n,m}(t)}{\mathrm{d}t^2}\ \sin\left(\frac{n\ \pi}{l_1}\ x\right)\sin\left(\frac{m\ \pi}{l_2}\ y\right) +$$

$$c^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{n^2 \pi^2}{l_1^2} + \frac{m^2 \pi^2}{l_2^2} \right) a_{n,m}(t) \sin\left(\frac{n \pi}{l_1} x \right) \sin\left(\frac{m \pi}{l_2} y \right) = 0$$

or

$$\frac{d^2 a_{n,m}(t)}{dt^2} + c^2 \left(\frac{n^2 \pi^2}{l_1^2} + \frac{m^2 \pi^2}{l_2^2} \right) a_{n,m}(t) = 0$$

We can now solve the problem by solving the family of ODEs

$$\frac{d^2 a_{n,m}(t)}{dt^2} + c^2 \left(\frac{n^2 \pi^2}{l_1^2} + \frac{m^2 \pi^2}{l_2^2} \right) a_{n,m}(t) = 0$$

$$a_{n,m}(0) = c_{n,m}$$

$$\frac{d a_{n,m}}{dt}(0) = 0$$

Neumann Boundary Conditions

Consider now the Poisson equation

$$- \Delta\, u(x,y) = f(x,y)$$

on a rectangular spacial region $\Omega = \{x,y|\ 0 \le x \le l_1\ ,\ 0 \le y \le l_2\}$ with Neumann boundary conditions:

$$u \in \{\ v \in C^2[\Omega] \mid \frac{\partial v}{\partial x}(0,y) = \frac{\partial v}{\partial x}(l_1,y) = 0,\ \frac{\partial v}{\partial y}(x,0) = \frac{\partial v}{\partial y}(x,l_2) = 0\ \}$$

An application of the divergence theorem gives us that

$$\int_\Omega f(\mathbf{x})\, d\mathbf{x} = - \int_\Omega \Delta\, u(\mathbf{x})\, d\mathbf{x} = - \int_\Omega \nabla \cdot \nabla u(\mathbf{x})\, d\mathbf{x} = - \int_{\delta\Omega} \nabla u(\mathbf{x}) \cdot \mathbf{n}\, \mathbf{d}\mathbf{x} = -\int_{\delta\Omega} \frac{\partial u}{\partial \mathbf{n}}\, \mathbf{d}\mathbf{x} = 0$$

which leads to a compatibility condition for this problem.

$$\int_\Omega f(\mathbf{x})\, d\mathbf{x} = 0$$

To solve this problem, we start by assuming once again that we can solve for eigenfunctions of the differential operator by the method of separation of variables.

$$u(x,y) = u_1(x)\ u_2(y)$$

Following reasoning similar to that we applied above, we see again that the problem separates into two sets of ODEs for $u_1(x)$ and $u_2(x)$.

The first ODE

$$\frac{d^2 u_1(x)}{dx^2} + \theta_1\, u_1(x) = 0$$

$$\frac{du_1}{dx}(0) = \frac{du_1}{dx}(l_1) = 0$$

has solutions

$$u_1(x) = \cos\left(\frac{n\,\pi}{l_1}\, x\right)\,,\ n = 0,1,2,...$$

which leads again to

$$\theta_1 = \frac{n^2\,\pi^2}{l_1^2}$$

Likewise, the second ODE has solutions

$$u_2(y) = \cos\left(\frac{m\,\pi}{l_2}\, y\right)\,,\ m = 0,1,2,...$$

which leads again to

$$\theta_2 = \frac{m^2\,\pi^2}{l_2^2}$$

The novel aspect in this case is the presence of 0 eigenvalues, which slightly complicates the task of forming Fourier expansions. For example, to compute the Fourier coefficients of the forcing function $f(x,y)$ we have to consider the $n = 0$ and $m = 0$ cases separately.

The first case

$$c_{0,0} = \frac{2}{l_2}\,\frac{2}{l_1}\int_0^{l_2}\int_0^{l_1} f(x,y)\; dx\, dy$$

will vanish since $f(x,y)$ has to satisfy the compatibility condition.

The next cases are

$$c_{0,m} = \frac{2}{l_2}\,\frac{2}{l_1}\int_0^{l_2}\int_0^{l_1} f(x,y)\cos\left(\frac{m\,\pi}{l_2}\, y\right)\, dx\, dy$$

$$c_{n,0} = \frac{2}{l_2}\,\frac{2}{l_1}\int_0^{l_2}\int_0^{l_1} f(x,y)\cos\left(\frac{n\,\pi}{l_1}\, x\right)\, dx\, dy$$

and the general case is

$$c_{n,m} = \frac{2}{l_2}\,\frac{2}{l_1}\int_0^{l_2}\int_0^{l_1} f(x,y)\cos\left(\frac{n\,\pi}{l_1}\,x\right)\cos\left(\frac{m\,\pi}{l_2}\,y\right)\,dx\,dy$$

To solve the Poisson equation we now assume that the solution takes the form

$$u(x,y) = \sum_{n=1}^{\infty} a_{n,0}\,\cos\left(\frac{n\,\pi}{l_1}\,x\right) + \sum_{m=1}^{\infty} a_{0,m}\cos\left(\frac{m\,\pi}{l_2}\,y\right) + \sum_{m\,=\,1}^{\infty}\sum_{n\,=\,1}^{\infty} a_{n,m}\cos\left(\frac{n\,\pi}{l_1}\,x\right)\cos\left(\frac{m\,\pi}{l_2}\,y\right)$$

To solve for the coefficients $a_{n,m}$ we solve

$$\frac{n^2\,\pi^2}{l_1^{\,2}}\,a_{n,0} = c_{n,0}$$

$$\frac{m^2\,\pi^2}{l_2^{\,2}}\,a_{0,m} = c_{0,m}$$

$$\left(\frac{n^2\,\pi^2}{l_1^{\,2}} + \frac{m^2\,\pi^2}{l_2^{\,2}}\right)a_{n,m} = c_{n,m}$$