Mathematical Preliminaries

Let $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector-valued function.

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{bmatrix}$$

the functions $f_k(x_1,x_2,x_3)$ are the *component functions* of the vector-valued function $\mathbf{f}(\mathbf{x})$.

Recall from Calculus III the definition of the divergence of $\mathbf{f}(\mathbf{x})$, $\nabla \cdot \mathbf{f}(\mathbf{x})$

$$\nabla \cdot \mathbf{f}(\mathbf{x}) = \frac{\partial f_1(x_1,x_2,x_3)}{\partial x_1} + \frac{\partial f_2(x_1,x_2,x_3)}{\partial x_2} + \frac{\partial f_3(x_1,x_2,x_3)}{\partial x_3}$$

The central result involving the divergence is the *divergence theorem*, which says that if $\mathbf{f}(\mathbf{x})$ is a vector-valued function defined on some closed and bounded spatial region Ω with boundary $\delta\Omega$ then

$$\int_\Omega \nabla \cdot \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \int_{\delta\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{n}\, \mathbf{d}\mathbf{x}$$

If $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a real-valued function defined on $\mathbb{R}^3$ the *gradient* of f, $\nabla f(\mathbf{x})$ is defined by

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial f(x_1,x_2,x_3)}{\partial x_1} \\ \dfrac{\partial f(x_1,x_2,x_3)}{\partial x_2} \\ \dfrac{\partial f(x_1,x_2,x_3)}{\partial x_3} \end{bmatrix}$$

The *directional derivative* of the real-valued function $f(\mathbf{x})$ in the direction of the vector $\mathbf{n}$ is

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{n}} = \nabla f(\mathbf{x}) \cdot \mathbf{n} = \frac{\partial f(x_1,x_2,x_3)}{\partial x_1}\, \mathbf{n}_1 + \frac{\partial f(x_1,x_2,x_3)}{\partial x_2}\, \mathbf{n}_2 + \frac{\partial f(x_1,x_2,x_3)}{\partial x_3}\, \mathbf{n}_3$$

The Laplacian of f, Δf is defined by

$$\Delta f(\mathbf{x}) = \nabla \cdot \nabla f(x) = \frac{\partial^2 f(x_1,x_2,x_3)}{\partial {x_1}^2} + \frac{\partial^2 f(x_1,x_2,x_3)}{\partial {x_2}^2} + \frac{\partial^2 f(x_1,x_2,x_3)}{\partial {x_3}^2}$$

The analogue for the integration by parts formula for real-valued functions on $\mathbb{R}^n$ is *Green's first identity*:

$$\int_\Omega v\, \Delta\, u\, d\mathbf{x} = \int_{\delta\Omega} v\, \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}}\, \mathbf{d}\mathbf{x} - \int_\Omega \nabla v \cdot \nabla u\, d\mathbf{x}$$

Deriving the Heat Equation in multiple spatial dimensions

Suppose we have a solid material occupying some region of space Ω. The heat equation is an equation that allows us to compute what the temperature $u(\mathbf{x},t)$ will be at each point in this region and all times t.

The first physical consideration that is relevant here is that temperature is proportional to heat content. In particular, if the material in question has a mass density of ρ and a heat capacity of c, the amount of heat energy found in some region Ω is given by

$$E(t) = \int_{\Omega} \rho \; c \; u(\mathbf{x},t) \, d\mathbf{x}$$

The heat equation concerns itself with the change in this energy content over time:

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = \int_{\Omega} \rho \; c \; \frac{\partial u(\mathbf{x},t)}{\partial t} \, d\mathbf{x}$$

This change in energy content will be driven by two factors, the flow of heat across the boundary $\delta\Omega$ and any external heat energy forcing function $f(\mathbf{x},t)$. If $\mathbf{q}(\mathbf{x},t)$ is the amount of heat flowing past some point $\mathbf{x}$ on the boundary $\delta\Omega$ at time t, the total heat flow across the boundary is given by

$$\int_{\delta\Omega} \mathbf{q}(\mathbf{x},t) \cdot (\text{-}\mathbf{n}) \, d\mathbf{x}$$

The amount of heat flow is governed by Fourier's law, which says that the heat flow past some point is proportional to the temperature gradient at that point.

$$\mathbf{q}(\mathbf{x},t) = \text{-} \; \kappa \; \nabla u(\mathbf{x},t)$$

Here κ is a physical constant that measures the heat conductivity of the material in question. Putting these two facts together gives us an expression for the heat flow past the boundary:

$$\int_{\delta\Omega} \mathbf{q}(\mathbf{x},t) \cdot (\text{-}\mathbf{n}) \, d\mathbf{x} = \int_{\delta\Omega} \text{-} \; \kappa \; \nabla u(\mathbf{x},t) \cdot (\text{-}\mathbf{n}) \, d\mathbf{x} = \kappa \int_{\delta\Omega} \nabla u(\mathbf{x},t) \cdot \mathbf{n} \; \mathbf{dx}$$

An application of the divergence theorem gives us that

$$\kappa \int_{\delta\Omega} \nabla u(\mathbf{x},t) \cdot \mathbf{n} \; \mathbf{dx} = \kappa \int_{\Omega} \nabla \cdot \nabla u(\mathbf{x},t) \, d\mathbf{x}$$

We now have the following expression for the energy balance in our space region Ω:

$$\int_{\Omega} \rho \; c \; \frac{\partial u(\mathbf{x},t)}{\partial t} \, d\mathbf{x} = \kappa \int_{\Omega} \nabla \cdot \nabla u(\mathbf{x},t) \, d\mathbf{x} + \int_{\Omega} f(\mathbf{x},t) \, d\mathbf{x}$$

Since this energy balance equation should be true for any spatial region, it must also be true at any point in space. This leads to the heat equation,

$$\rho \; c \; \frac{\partial u(\mathbf{x},t)}{\partial t} \text{ - } \kappa \; \nabla \cdot \nabla u(\mathbf{x},t) = f(\mathbf{x},t)$$

The heat equation is more commonly expressed in terms of the Laplace operator.

$$\rho\ c\ \frac{\partial u(\mathbf{x},t)}{\partial t} - \kappa\ \Delta u(\mathbf{x},t) = f(\mathbf{x},t)$$

Note finally that to make a complete problem we also need to specify boundary conditions. This means that at each point in the boundary region $\delta\Omega$ we need to specify either the temperature at that point at each time t (Dirichlet boundary conditions), or the directional derivative of the temperature in the direction of the normal to the boundary. For example, the problem with homogeneous Dirichlet conditions has

$$u(\mathbf{x},t) = 0 \text{ for all } \mathbf{x}\in\delta\Omega \text{ and all } t \geq 0$$

which means that the boundary is being maintained at a constant temperature of 0. The problem with homogeneous Neumann boundary conditions has

$$\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} = 0 \text{ for all } \mathbf{x}\in\delta\Omega \text{ and all } t \geq 0$$

The latter condition basically says that there is no heat flow across the boundary.

The Steady State Heat Equation

An interesting and important special case of the heat equation is the steady-state version. In this version the forcing function and the boundary conditions are both independent of t. In that situation, the system will eventually settle into an equilibrium temperature distribution that is itself independent of time. Since that equilibrium distribution is independent of time we have immediately that

$$\rho\ c\ \frac{\partial u(\mathbf{x},t)}{\partial t} = 0$$

and the equation reduces to either the *Poisson equation*

$$-\kappa\ \Delta u(\mathbf{x}) = f(\mathbf{x})$$

or the *Laplace equation*

$$-\kappa\ \Delta u(\mathbf{x}) = 0$$

in the case in which there is no external forcing function. These equations are also important in their own right, and have many applications beyond the steady-state heat distribution problem.

Symmetry and Eigenvalues of the Laplace Operator

The Poisson and Laplace equations both invite us to examine the Laplace operator

$$L\ u = -\Delta u$$

on some appropriate space of functions determined by a set of boundary conditions. For example, consider the Poisson equation with homogeneous Dirichlet boundary conditions:

$$L_D\ u(\mathbf{x}) = -\Delta u(\mathbf{x}) = f(\mathbf{x})$$

$$u(\mathbf{x}) = 0 \text{ for all } \mathbf{x}\in\delta\Omega$$

Here are some basic quesions about this operator:

1. Is the operator symmetric? That is, is

$$\int_\Omega (L_D u(\mathbf{x}))\, v(\mathbf{x})\, d\mathbf{x} = (L_D\ u,v) = (u,L_D\ v) = \int_\Omega u(\mathbf{x})(L_D\ v(\mathbf{x}))\, d\mathbf{x}$$

for all functions $u(\mathbf{x})$ and $v(\mathbf{x})$ satisfying the Dirichlet boundary conditions? The answer is yes: the proof relies on two applications of Green's first identity:

$$\int_\Omega v\ \Delta\ u\, d\mathbf{x} = \int_{\delta\Omega} v\ \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}}\, \mathbf{dx} - \int_\Omega \nabla v \cdot \nabla u\, d\mathbf{x}$$

$$(L_D\ u,v) = \int_\Omega (-\Delta u(\mathbf{x}))\, v(\mathbf{x})\, d\mathbf{x}$$

$$= \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})\, d\mathbf{x} - \int_{\delta\Omega} v(\mathbf{x})\ \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}}\, \mathbf{dx}$$

$$= \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})\, d\mathbf{x}$$

$$= \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})\, d\mathbf{x} - \int_{\delta\Omega} u(\mathbf{x})\ \frac{\partial v(\mathbf{x})}{\partial \mathbf{n}}\, \mathbf{dx}$$

$$= -\int_\Omega u(\mathbf{x}) \Delta v(\mathbf{x})\, d\mathbf{x} = (u,L_D\ v)$$

2. Are the eigenvalues strictly positive? Suppose $u(x)$ is an eigenfunction of L_D with $(u,u) = 1$.

$$\lambda = \lambda(u,u) = (\lambda u,u) = (L_D u,u) = -\int_\Omega (-\Delta u(\mathbf{x}))\, u(\mathbf{x})\, d\mathbf{x}$$

$$= \int_\Omega \nabla u(\mathbf{x}) \nabla u(\mathbf{x})\, d\mathbf{x} - \int_{\delta\Omega} u(\mathbf{x})\ \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}}\, \mathbf{dx}$$

$$= \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x})\, d\mathbf{x}$$

This latter integral is strictly greater than 0, because the only way for it to be 0 would be for $u(\mathbf{x})$ to be a constant function whose gradient everywhere is $\mathbf{0}$. The only constant functions that satisfy the Dirichlet boundary conditions are $u(\mathbf{x}) \equiv 0$. This is a contradiction, because we assumed at start that $(u,u) = 1$.