Expressing Systems as Matrix Equations

Consider a system of n linear equations in n unknowns.

$$\left\{ \begin{array}{c} a_{1,1}\, x_1 + a_{1,2}\, x_2 + \cdots + a_{1,n}\, x_n = b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 + \cdots + a_{2,n}\, x_n = b_2 \\ \vdots \\ a_{m,1}\, x_1 + a_{m,2}\, x_2 + \cdots + a_{m,n}\, x_n = b_m \end{array} \right\}$$

It is possible to rewrite this system as a matrix equation.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or more compactly

$$A\, \vec{x} = \vec{b}$$

Conceptually, the simplest way to solve this equation is to multiply both sides by the multiplicative inverse of A, A^{-1}.

$$\vec{x} = A^{-1}\, \vec{b}$$

What is the inverse of a matrix?

The ordinary multiplicative inverse of a number a is a number a^{-1} with the property that

$$a\, a^{-1} = a^{-1}\, a = 1$$

By analogy, the multiplicative inverse of a matrix A should be a matrix A^{-1} with the property that

$$A\, A^{-1} = A^{-1}\, A = I$$

Because matrix multiplication is not commutative, we may have to allow for the possibility that a matrix may have both a *left inverse* and a *right inverse*. The left inverse of a matrix A is a matrix B with the property that

$$B\, A = I$$

The right inverse is a matrix C with the property that

$$A\, C = I$$

In many cases it may well happen that B and C are the same matrix. In that case, we refer to that matrix as simply the inverse of A and call it A^{-1}.

Beyond the left/right inverse confusion there are some other problems.

1. How can we tell whether or not a given matrix has an inverse?
2. How can we compute the inverse?
3. Is the inverse unique, or can a matrix have more than one inverse?

The key to most of these problems is to find a way to compute the inverse.

Gauss Elimination and Matrix Inversion

Suppose we are trying to compute the right inverse of a matrix A. We seek a matrix C such that

$$A\ C = I$$

For simplicity, let us assume that all three matrices involved are n by n matrices. For non-square matrices the argument is similar, so it should suffice to study the square case. What we are looking for is a matrix C such that

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

An important observation about the way that matrix multiplication works is that the j^{th} column of a product of two matrices is the same as the product of the first matrix times the j^{th} column of the second matrix. In other words, the matrix equation above breaks into n distinct vector equations:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} c_{1,1} \\ c_{2,1} \\ \vdots \\ c_{n,1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} c_{1,2} \\ c_{2,2} \\ \vdots \\ c_{n,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} c_{1,n} \\ c_{2,n} \\ \vdots \\ c_{n,n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Note that these n systems can be solved by Gauss elimination. This gives us a way to solve for the n columns of C and ultimately for C itself.

A further useful observation about the above n problems is that when we set out to solve these systems we will be doing the same row operations on each of the n systems. The reason for this is that the coefficient matrix of each of these systems is the same, so the steps needed to reduce the coefficient matrix to a diagonal matrix will be the same in every case. The only thing that varies in each case is the last column of the augmented matrix, which comes from the right hand side of the system in question. This suggests that we can solve these n systems in parallel by writing down the coefficient matrix just once and working on all n of the right hand sides in parallel. A simple and practical way to do this is to construct a sort of super-augmented matrix with A on the left and the n right hand side columns on the right:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & 1 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Or more compactly,

$$[A \mid I\,]$$

If we do a complete Gauss-Jordan elimination on this super-augmented matrix that reduces the coefficient matrix A all the way down to the identity I, the columns that will appear on the right will be the n solutions of the n systems shown above. Note that those n solutions just happen to be the n columns of C:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ 0 & 1 & \cdots & 0 & c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix}$$

This now gives us a practical algorithm for computing an inverse based on Gauss-Jordan elimination.

Some theorems

Technically, the above method only shows us how to compute a right inverse C for a matrix A. A little additional theory will clear things up completely.

Definition An n by n matrix A is *non-singular* if there exists a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

A matrix A is *singular* if it has no inverse.

Theorem If an n by n matrix A has a right inverse C then the left inverse $B = C$ and $B = C = A^{-1}$.

Proof If B is a left inverse for A then

$$B\,A = I$$

Multiplying both sides on the right by the right inverse C gives

$$B\,A\,C = C$$

$$B\,(A\,C) = C$$

$$B\,(I) = C$$

$$B = C$$

Corollary If a square matrix A can be reduced all the way to the identity via Gauss-Jordan elimination then A is non-singular. Likewise, if a row of 0s appears in the course of Gauss-Jordan elimination then A is a singular matrix and does not have an inverse.