Differential Equations

Our concern in chapter five is solving differential equations numerically. A *differential equation* is an equation involving some unknown function $y(t)$ and its derivatives. The simplest class of differential equations we will study is the *first order* equation.

$$y'(t) = f(t,y(t))$$

where $f(t,y)$ is some known function of t and y. To solve this equation we will also need an additional piece of information, an *initial value* for the function $y(t)$.

$$y(t_0) = \alpha$$

Some notation

All of the methods we will see in chapter 5 attempt to compute estimates for $y(t_k) = y_k$. The only y_k we will be able to compute exactly is of course $y_0 = y(t_0) = \alpha$ which is given to us as an intitial condition.

All of the methods in chapter 5 work by computing a sequence w_k of approximations for the actual y_k points. In every case, the point w_{k+1} in the sequence is computed from w_k, t_k, and the step size $h = t_{k+1} - t_k$. To emphasize this dependence we write

$$w_{k+1} = w_k + h\,\varphi(t_k,w_k)$$

One of our primary concerns when generating this sequence of w_k values is the size of the error we are making. One way to measure this error is to ask what the size of $|y_{k+1} - w_{k+1}|$ is if we assume that $w_k = y_k$. That is, assuming that our w value is exact at some point, how much of an error will one step introduce?

This is not quite the correct measure to use, because we can always artificially shrink $|y_{k+1} - w_{k+1}|$ by simply shrinking the step size h. The more relevant question to ask is at what rate $|y_{k+1} - w_{k+1}|$ grows as a function of h. To measure that rate we introduce the *truncation error* $\tau_{k+1}(h)$:

$$\tau_{k+1}(h) = \frac{y_{k+1} - w_{k+1}}{h} = \frac{y_{k+1} - (y_k + h\,\varphi(t_k,w_k))}{h} = \frac{y_{k+1} - y_k}{h} - \varphi(t_k,w_k)$$

An important thing to note about the truncation error is that this is the *local error* incurred in a single step of the method. The methods we study in chapter 5 are usually iterated through many steps. Although it is generally possible to produce estimates of the error incurred in a single step, estimating the cumulative effect of these errors as they compound over multiple iterations is much harder to do. For our purposes, we will focus on the local truncation error since it is usually easy to estimate and control.

Methods based on Taylor's Theorem

If we know the value of $y(t)$ at $t = t_k$ and we have some information about the derivatives of $y(t)$ we can use Taylor's theorem to construct an estimate of the value of the function at some nearby point $t = t_{k+1}$:

$$y(t_{k+1}) \approx y(t_k) + y'(t_k)\,(t_{k+1} - t_k) + \frac{y''(t_k)}{2}\left(t_{k+1} - t_k\right)^2 + \cdots + \frac{y^{(n)}(t_k)}{n!}\left(t_{k+1} - t_k\right)^n$$

The more commonly used form of this equation takes account of the spacing between t_k and t_{k+1}: we write

$$t_{k+1} = t_k + h$$

and express the expansion as

$$y(t_k + h) \approx y(t_k) + y'(t_k)\,h + \cdots + \frac{y^{(n)}(t_k)}{n!}\,h^n$$

How can we compute the necessary derivative terms? The first term can be computed from the differential equation itself:

$$y'(t_k) = f(t_k, y(t_k)) = f(t_k, y_k)$$

Higher derivatives will require some additional work. We will see below how to handle those derivatives.

Euler's method

The simplest Taylor method is the Taylor method of order one, also known as Euler's method. In this method we expand $y(t)$ to only the first derivative around t_k and use the differential equation to express the first derivative in terms of $f(t,y)$:

$$y(t_k + h) \approx y(t_k) + y'(t_k)\,h = y(t_k) + h\,f(t_k, y(t_k))$$

Expressed in terms of our notational framework, this leads to an interative method

$$w_{k+1} = w_k + h\,f(t_k, w_k) = w_k + h\,\varphi(t_k, w_k)$$

As we will see when we discuss error estimates below, this is not a very accurate method.

Higher order Taylor methods

The more accurate approach is to use more terms in the Taylor expansion.

$$y(t_k + h) \approx y(t_k) + y'(t_k)\,h + \cdots + \frac{y^{(n)}(t_k)}{n!}\,h^n$$

To compute the higher order derivatives we can differentiate the differential equation. For example, to compute the second derivative term we would do

$$y''(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t, y(t)) = \frac{\partial f(t,y)}{\partial t} + \frac{\partial f(t,y)}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial f(t,y)}{\partial t} + \frac{\partial f(t,y)}{\partial y} f(t,y)$$

In practice, this is most easily done by writing out $f(t, y(t))$ directly and just differentiating. Here is a concrete example. Consider the differential equation

$$y'(t) = t^2 + y^2(t)$$

Differentiating this with respect to t gives

$$y''(t) = 2\,t + 2\,y(t)\,y'(t) = 2\,t + 2\,y(t)\left(t^2 + y^2(t)\right)$$

The third derivative term is

$$y^{(3)}(t) = 2 + 2\,y'(t)\left(t^2 + y^2(t)\right) + 2\,y(t)\left(2\,t + 2\,y(t)\,y'(t)\right)$$

$$= 2 + 4\,t\,y(t) + \left(2\,t^2 + 6\,y^2(t)\right)y'(t)$$

$$= 2 + 4\,t\,y(t) + \left(2\,t^2 + 6\,y^2(t)\right)\left(t^2 + y^2(t)\right)$$

Since all of these expressions compute the desired derivatives of $y(t)$ in terms of t and $y(t)$, we can evaluate all of these expressions at $t = t_k$ using $y(t_k) = y_k$.

As you can see here, the problem with this approach is that computing the higher order derivatives is a real chore. The accompanying Mathematica notes will show how to foist the necessary symbolic manipulation chores off onto Mathematica.

More notation

We can place the Taylor method

$$y(t_k + h) \approx y(t_k) + y'(t_k)\,h + \cdots + \frac{y^{(n)}(t_k)}{n!}\,h^n$$

into our standard notational framework by rewriting it as

$$w_{k+1} = w_k + h\,(y'(t_k) + \frac{y''(t_k)}{2}\,h + \cdots + \frac{y^{(n)}(t_k)}{n!}\,h^{n-1})$$

and noting that

$$\varphi(t_k,w_k) = y'(t_k) + \frac{y''(t_k)}{2}\,h + \cdots + \frac{y^{(n)}(t_k)}{n!}\,h^{n-1}$$

To emphasize the fact that this particular φ comes from Taylor's expansion, we introduce

$$\varphi(t_k,w_k) = T^{(n)}(t_k,w_k) = y'(t_k) + \frac{y''(t_k)}{2}\,h + \cdots + \frac{y^{(n)}(t_k)}{n!}\,h^{n-1}$$

Error estimates

Error estimates for the method above come directly from the error term in Taylor's theorem.

$$y(t_k + h) = y(t_k) + y'(t_k)\,h + \cdots + \frac{y^{(n)}(t_k)}{n!}\,h^n + \frac{y^{(n+1)}(\xi(t_k,h))}{(n+1)!}\,h^{n+1}$$

Using our standardized notation from above, this can also be written

$$w_{k+1} = w_k + h\, T^{(n)}(t_k, w_k)$$

$$y_{k+1} = w_{k+1} + \frac{y^{(n+1)}(\xi(t_k, h))}{(n+1)!}\, h^{n+1}$$

Using the definition for the truncation error we find

$$\tau_{k+1}(h) = \frac{y_{k+1} - w_{k+1}}{h} = \frac{y^{(n+1)}(\xi(t_k, h))}{(n+1)!}\, h^n$$

Assuming that the $(n+1)^{st}$ derivative of $y(t)$ is bounded, we see that the truncation error for the n^{th} order Taylor method is $O(h^n)$.

Since Euler's method is a Taylor method of order $n = 1$, we see that the truncation error for Euler's method is $O(h)$
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