Systems of Equations

A system of first order differential equations is a list of m coupled first-order differential equations in m unknown functions of the independent variable t along with an appropriate set of initial conditions.

$$\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} = f_1(t, u_1(t), u_2(t), \dots, u_m(t))$$

$$\frac{\mathrm{d}u_2(t)}{\mathrm{d}t} = f_2(t, u_1(t), u_2(t), \dots, u_m(t))$$

$$\vdots$$

$$\frac{\mathrm{d}u_m(t)}{\mathrm{d}t} = f_m(t, u_1(t), u_2(t), \dots, u_m(t))$$

$$u_1(t_0) = \alpha_1$$

$$u_2(t_0) = \alpha_2$$

$$\vdots$$

$$u_m(t) = \alpha_m$$

It is important to note that these equations are coupled: the right hand side of each equation is allowed to depend on all of the unknown functions $u_1(t)$, $u_2(t)$, …, $u_m(t)$.

The most straightforward to approach the solution of these systems is to recast the problem in terms of vector-valued functions. Specifically, we introduce a vector-valued function

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

and a vector-valued function

$$\mathbf{f}(t, \mathbf{u}(t)) = \begin{bmatrix} f_1(t, \mathbf{u}(t)) \\ f_2(t, \mathbf{u}(t)) \\ \vdots \\ f_m(t, \mathbf{u}(t)) \end{bmatrix}$$

and write our system in vector form:

$$\frac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{u}(t))$$

$$\mathbf{u}(t_0) = \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}$$

The advantage to rewriting the system in this form is that it strongly suggests that we can treat this problem with the same techniques we used earlier for solving first order equations in one unknown function.

This works, because all of the operations required to implement the methods we developed earlier are operations that extend in a natural way to vector-valued functions.

Here is an example. Here is the equation that governs the midpoint method we developed in section 5.4.

$$w_{k+1} = w_k + h\, f(t_k + \frac{h}{2}\, , w_k + \frac{h}{2} f(t_k,w_k))$$

This equation generalizes in a natural way to a vector system.

$$\mathbf{w}_{k+1} = \mathbf{w}_k + h\, \mathbf{f}(t_k + \frac{h}{2}\, , \mathbf{w}_k + \frac{h}{2} f(t_k,\mathbf{w}_k))$$

An example

The best way to demonstrates that this works is to show a simple example.

$$u_1'(t) = -4\, u_1(t) - 2\, u_2(t) + \cos t + 4 \sin t$$

$$u_2'(t) = 3\, u_1(t) + u_2(t) - 3 \sin t$$

$$u_1(0) = 0$$

$$u_2(0) = -1$$

In vector form this is

$$\mathbf{u}'(t) = \mathbf{f}(t,\mathbf{u}(t))$$

$$\mathbf{u}(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

where

$$\mathbf{f}(t,\mathbf{u}(t)) = \begin{bmatrix} -4\, u_1(t) - 2\, u_2(t) + \cos t + 4 \sin t \\ 3\, u_1(t) + u_2(t) - 3 \sin t \end{bmatrix}$$

The midpoint method for this example is

$$\mathbf{w}_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\mathbf{z}_k = f(t_k,\mathbf{w}_k) = \begin{bmatrix} -4\, w_{k,1} - 2\, w_{k,2} + \cos t_k + 4 \sin t_k \\ 3\, w_{k,1} + w_{k,2} - 3 \sin t_k \end{bmatrix}$$

$$\mathbf{w}_{k+1} = \mathbf{w}_k + h\, \mathbf{f}(t_k + \frac{h}{2}, \mathbf{w}_k + \frac{h}{2} \mathbf{z}_k)$$

$$= \begin{bmatrix} w_{k,1} \\ w_{k,2} \end{bmatrix} + h\, \mathbf{f}(t_k + \frac{h}{2}, \begin{bmatrix} w_{k,1} \\ w_{k,2} \end{bmatrix} + \frac{h}{2} \begin{bmatrix} z_{k,1} \\ z_{k,2} \end{bmatrix})$$

$$= \begin{bmatrix} w_{k,1} \\ w_{k,2} \end{bmatrix} + h \begin{bmatrix} -4\left(w_{k,1} + \frac{h}{2} z_{k,1}\right) - 2\left(w_{k,2} + \frac{h}{2} z_{k,2}\right) + \cos\left(t_k + \frac{h}{2}\right) + 4 \sin\left(t_k + \frac{h}{2}\right) \\ 3\left(w_{k,1} + \frac{h}{2} z_{k,1}\right) + \left(w_{k,2} + \frac{h}{2} z_{k,2}\right) - 3 \sin\left(t_k + \frac{h}{2}\right) \end{bmatrix}$$

This is a situation where Mathematica can handle the symbolic manipulation for us quite well.

Multi-stage methods

All multi-stage methods generalize in a straightforward way to systems. In particular, the standard Runge-Kutta method of order four

$$\mathbf{k}_1 = \mathbf{f}(t_k,\mathbf{w}_k)$$

$$\mathbf{k}_2 = \mathbf{f}(t_k + h/2,\mathbf{w}_k + h/2\, \mathbf{k}_1)$$

$$\mathbf{k}_3 = \mathbf{f}(t_k + h/2,\mathbf{w}_k + h/2\, \mathbf{k}_2)$$

$$\mathbf{k}_4 = \mathbf{f}(t_k + h,\mathbf{w}_k + h\, \mathbf{k}_3)$$

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \frac{h}{3}\left(\mathbf{k}_1 + 2\, \mathbf{k}_2 + 2\, \mathbf{k}_3 + \mathbf{k}_4\right)$$

carries over to systems quite readily.

Higher order equations

A second order differential equation takes the form

$$y''(t) = f(t,y(t),y'(t))$$

$$y(t_0) = \alpha$$

$$y'(t_0) = \beta$$

This equation can be converted to a system of two first order equations by introducing functions $u_1(t) = y(t)$ and $u_2(t) = y'(t)$. This converts the equation above into a system of equations

$$\mathbf{u}'(t) = \begin{bmatrix} u_2(t) \\ u_1(t) \end{bmatrix}' = \begin{bmatrix} f(t, u_1(t), u_2(t)) \\ u_2(t) \end{bmatrix} = \mathbf{f}(t, \mathbf{u}(t))$$

$$\mathbf{u}(t_0) = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

which can then be solved by any of the methods described above. This method generalizes to differential equations of any order.