Euler's method

In section 5.3 we saw the Taylor method for estimating solutions to differential equations. This method is based on doing a Taylor expansion of the solution function about a known point:

$$y(t_{k+1}) \approx y(t_k) + y'(t_k)\,(t_{k+1} - t_k) + \frac{y''(t_k)}{2}\,\left(t_{k+1} - t_k\right)^2 + \cdots + \frac{y^{(n)}(t_k)}{n!}\left(t_{k+1} - t_k\right)^n$$

In cases where the spacing between sample points is fixed at h, this simplifies to

$$y(t_{k+1}) \approx y(t_k) + y'(t_k)\,h + \frac{y''(t_k)}{2}\,h^2 + \cdots + \frac{y^{(n)}(t_k)}{n!}h^n$$

The simplest version of this method corresponds to setting $n = 1$:

$$y(t_{k+1}) \approx y(t_k) + y'(t_k)\,h$$

After substituting what we know about $y'(t)$ from the differential equation and using our standard notation for approximate solutions, we arrive at

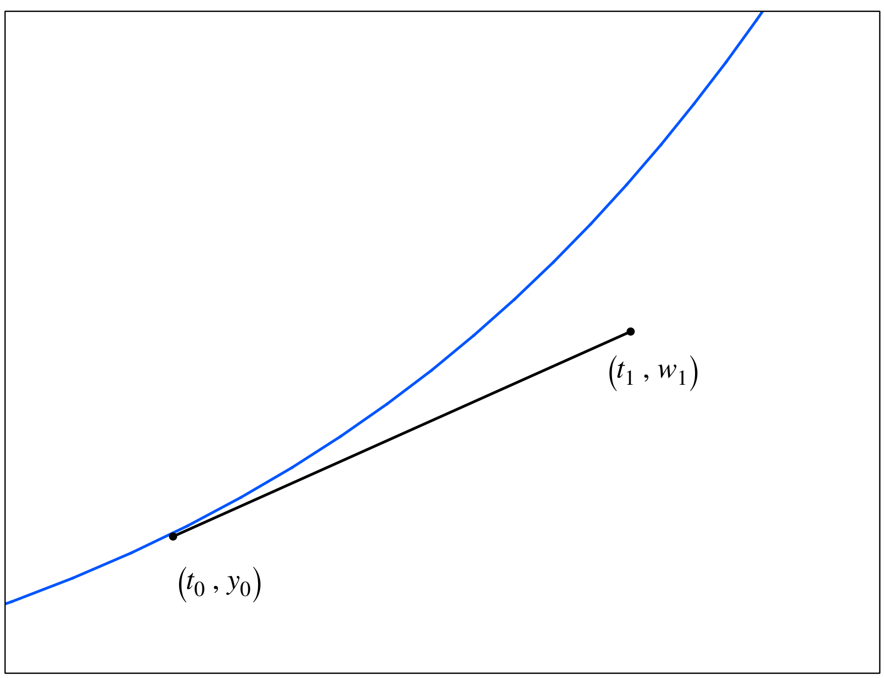
$$w_{k+1} = w_k + f(t_k, w_k)\,h$$

$$w_0 = y_0 = \alpha$$

This simplest form of the Taylor method is also known as Euler's method.

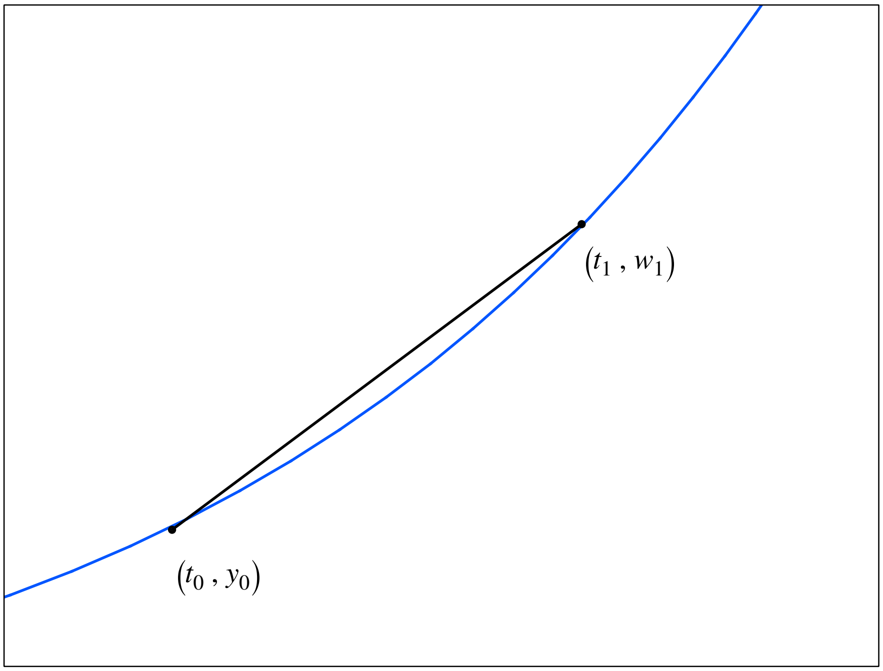
A multistep method

The picture below illustrates why Euler's method typically does not produce great results.



The method essentially constructs a tangent line to the solution curve at (t_k,y_k) and uses that construct an estimate at a later time. The problem with this approach is that the actual solution will not be travelling in a straight line. If we try to approximate it with a straight line we will be making an obvious error.

One potential fix is to replace the slope of the straight line with a better slope. Right now we are using $f(t_k,y_k)$ as our slope. If we were able to replace this with a better slope, we could get a better result.



This is the aim of so-called *multi-step methods*. The simplest multi-step method is the modified Euler method. In this method we compute a pair of slopes and average them together in an effort to get a better slope:

$$p_0 = w_k$$

$$k_1 = f(t_0,p_0)$$

$$p_1 = p_0 + k_1\, h$$

$$k_2 = f(t_1,p_1)$$

$$w_{k+1} = w_k + \frac{k_1 + k_2}{2}\, h$$

The general procedure in a multi-step method is to construct a set of slope estimates k_1 through k_n and then average them together to make the slope we will use to compute w_{k+1} from w_k.

$$k_1 = f(t_k,w_k)$$

$$k_i = f\left(t_k + \alpha_i\, h\, ,\, w_k + h \sum_{j=1}^{i-1} \beta_{i,j}\, k_j \right)$$

$$s = \sum_{i=1}^{n} \gamma_i \, k_i$$

$$w_{k+1} = w_k + s\,h$$

This approach has two advantages. One is that the necessary calculations are simpler than the calculations we would have to do in a Taylor method. The second advantage is that this method has a lot of constants we can tweak, so we can potentially tune this method to produce results that are as good as a Taylor method of degree n. The Mathematica notebook I have provided for chapter five works through some examples to illustrate how this can be done.