Applying the Richardson extrapolation to the composite trapezoid rule

In the composite trapezoid rule we take an interval of the form $[a,b]$ and slice it into n equal-sized subintervals of width $h = (b\text{-}a)/n$ and then use the trapezoid rule on each subinterval.

$$\int_a^b f(x)\,\mathrm{d}\,x = \sum_{i=0}^{n-1} h \frac{f(x_i) + f(x_{i+1})}{2} + O(h^2)$$

Here $x_i = a + i\,h$. The error term in the composite trapezoid rule has a term proportional to h^2 as its leading term.

To apply the Richardson extrapolation to this method we start by abstracting away many of the details of the calculation, and also expand the error term to capture more of its structure:

$$\int_a^b f(x)\,\mathrm{d}\,x = sum(h) + c_1\,h^2 + c_2\,h^3 + \cdots$$

To apply the Richardson extrapolation we write down two versions of this equality using different values of h and then combine those two versions algebraically in an effort to remove the largest term in the error estimate:

$$\int_a^b f(x)\,\mathrm{d}\,x = sum(h) + c_1\,h^2 + c_2\,h^4 + \cdots$$

$$4\int_a^b f(x)\,\mathrm{d}\,x = 4\,sum(h/2) + 4\,c_1\left(\frac{h}{2}\right)^2 + 4\,c_2\left(\frac{h}{2}\right)^4 + \cdots$$

Subtracting these two equations gives

$$3\int_a^b f(x)\,\mathrm{d}\,x = 4\,sum(h/2) - sum(h) - \frac{3}{4}\,c_2\,h^4 + \cdots$$

or

$$\int_a^b f(x)\,\mathrm{d}\,x = \frac{4\,sum(h/2) - sum(h)}{3} + \frac{1}{4}\,c_2\,h^4 + \cdots$$

This yeilds an $O(h^4)$ estimate for the integral, which is a noticable improvement over the original $O(h^2)$ trapezoid rule estimate.

This process can then be repeated. In the next round of the extrapolation we write down two versions of this last formula and combine them to get an even better estimate.

This process can be summarized by introducing some additional notation. We introduce the new notation

$$R[k,0] = sum((b\text{-}a)/2^k)$$

Successive rounds of the Richardson extrapoltation can then be written

$$R[k,j] = R[k,j\text{-}1]+\frac{R[k,j\text{-}1] - R[k\text{-}1,j\text{-}1]}{4^j\text{-}1}$$

The error term for $R[k,j]$ has magnitude $O\left(\left((b\text{-}a)/2^k\right)^{2+2j}\right)$.