Improving on interpolation

In earlier sections we saw several techniques for interpolating a function. Given a set of n+1 data points $(x_0, f(x_0))$, $(x_1, f(x_1))$, ... , $(x_n, f(x_n))$ we can construct an n^{th} degree polynomial that interpolates those points.

One limitation of this scheme is that the polynomial we construct is only constrained to agree with the function at the n+1 sample points. In principle, we could produce better results if we could force the polynomial and its first derivative to agree with the value of f(x) and its first derivative $f'(x)$ at a series of sample points. The technique of Hermite polynomials accomplishes this goal.

Here is a theorem that shows how this is possible.

Theorem (Hermite Interpolation) If $f \in C^1[a,b]$ and $x_0, x_1, \ldots, x_n \in [a,b]$ are distinct, the unique polynomial of least degree agreeing with f and f' at $x_0, x_1, \ldots, x_n$ is the Hermite polynomial of degree at most $2\,n + 1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)\, H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\, \hat{H}_{n,j}(x)$$

where, for $L_{n,j}(x)$ denoting the j^{th} Lagrange basis function for the given data points, we have

$$H_{n,j}(x) = [1 - 2\,(x - x_j)\, L'_{n,j}(x_j)]\, L^2_{n,j}(x)$$

$$\hat{H}_{n,j}(x) = (x - x_j)\, L^2_{n,j}(x)$$

Moreover, if $f \in C^{2n+2}[a,b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 (x - x_1)^2 \cdots (x - x_n)^2}{(2\,n + 2)!} f^{(2n+2)}(\xi(x))$$

for some $\xi(x) \in [a,b]$.

Proof Just as the basis functions were constructed to vanish at each interpolation point except one, the functions $H_{n,j}(x)$ and $\hat{H}_{n,j}(x)$ are constructed to have special behavior at the interpolation points. Specifically, if $i \neq j$ we have

$$H_{n,j}(x_i) = [1 - 2\,(x_i - x_j)\, L'_{n,j}(x_j)]\, L^2_{n,j}(x_i) = 0$$

$$\hat{H}_{n,j}(x_i) = (x_i - x_j)\, L^2_{n,j}(x_i) = 0$$

Further, since the derivatives

$$\left(H_{n,j}\right)'(x) = [-\,2\, L'_{n,j}(x_j)\,]\, L^2_{n,j}(x) + [1 - 2\,(x - x_j)\, L'_{n,j}(x_j)]\; 2\, L_{n,j}(x)\, L'_{n,j}(x)$$

$$\left(\hat{H}_{n,j}\right)'(x) = L^2_{n,j}(x) + (x - x_j)\; 2\, L_{n,j}(x)\, L'_{n,j}(x)$$

satisfy

$$\left(H_{n,j}\right)'(x_i) = [-\,2\, L'_{n,j}(x_j)\,]\, L^2_{n,j}(x_i) + [1 - 2\,(x_i - x_j)\, L'_{n,j}(x_j)]\; 2\, L_{n,j}(x_i)\, L'_{n,j}(x_i) = 0 + 0$$

$$\left(\hat{H}_{n,j}\right)'(x_i) = L^2_{n,j}(x_i) + (x_i - x_j)\, 2\, L_{n,j}(x_i)\, L'_{n,j}(x_i) = 0 + 0$$

For $i = j$ we have

$$H_{n,j}(x_j) = [1 - 2\,(x_j - x_j)\, L'_{n,j}(x_j)]\, L^2_{n,j}(x_j) = 1\ 1 = 1$$

$$\hat{H}_{n,j}(x_j) = (x_j - x_j)\, L^2_{n,j}(x_j) = 0\ 1 = 0$$

and

$$\left(H_{n,j}\right)'(x_j) = [- 2\, L'_{n,j}(x_j)\ ]\, L^2_{n,j}(x_j) + [1 - 2\,(x_j - x_j)\, L'_{n,j}(x_j)]\, 2\, L_{n,j}(x_j)\, L'_{n,j}(x_j)$$

$$= - 2\, L'_{n,j}(x_j) + 2\, L'_{n,j}(x_j) = 0$$

$$\left(\hat{H}_{n,j}\right)'(x_j) = L^2_{n,j}(x_j) + (x_j - x_j)\, 2\, L_{n,j}(x_j)\, L'_{n,j}(x_j) = 1$$

To summarize,

	$i \neq j$	$i = j$
$H_{n,j}(x_i)$	0	1
$\hat{H}_{n,j}(x_i)$	0	0
$\left(H_{n,j}\right)'(x_i)$	0	0
$\left(\hat{H}_{n,j}\right)'(x_i)$	0	1

Thus

$$H_{2n+1}(x_i) = \sum_{j=0}^{n} f(x_j)\, H_{n,j}(x_i) + \sum_{j=0}^{n} f'(x_j)\, \hat{H}_{n,j}(x_i) = f(x_i)$$

and

$$\left(H_{2n+1}\right)'(x_i) = \sum_{j=0}^{n} f(x_j)\, \left(H_{n,j}\right)'(x_i) + \sum_{j=0}^{n} f'(x_j)\, \left(\hat{H}_{n,j}\right)'(x_i) = f'(x_i)$$

The error formula part of the proof is an exercise.

Computing the Hermite Polynomials

Just as the original formula for Lagrange polynomials proved too unwieldy to use and had to be replaced with Neville's method and the Newton formulas, we also want to replace the clunky formulas above with something better.

This method is based on the observation that for any one of the interpolation points x_i we have

$$\lim_{\varepsilon \to 0} \frac{f(x_i + \varepsilon) - f(x_i)}{\varepsilon} = f'(x_i)$$

Expressed in the language of divided differences, this says

$$\lim_{\varepsilon \to 0} f\,[x_i, x_i + \varepsilon] = f'(x_i)$$

What would happen if we tried to use the usual Newton forward difference method to interpolate $f(x)$ at sample points $(x_0, f(x_0))$, $(x_0+\varepsilon, f(x_0+\varepsilon))$, $(x_1, f(x_1))$, $(x_1+\varepsilon, f(x_1+\varepsilon))$, ... , $(x_n, f(x_n))$, $(x_n+\varepsilon, f(x_n+\varepsilon))$?

First of all, since we have more sample points, we should expect to produce a polynomial of higher degree: degree $2\,n + 1$ to be exact. To construct the polynomial we would use a forward difference table:

x	$f(x)$	First diff.	Second diff.
x_0	$f[x_0] = f(x_0)$		
		$f[x_0, x_0+\varepsilon]$	
$x_0 + \varepsilon$	$f[x_0+\varepsilon] = f(x_0+\varepsilon)$		$f[x_0, x_0+\varepsilon, x_1] = \frac{f[x_0+\varepsilon, x_1] - f[x_0, x_0+\varepsilon]}{x_1 - x_0}$
		$f[x_0+\varepsilon, x_1]$	
x_1	$f[x_1] = f(x_1)$		$f[x_0+\varepsilon, x_1, x_1+\varepsilon] = \frac{f[x_1, x_1+\varepsilon] - f[x_0+\varepsilon, x_1]}{x_1 - x_0}$
		$f[x_1, x_1+\varepsilon]$	
$x_1 + \varepsilon$	$f[x_1+\varepsilon] = f(x_1+\varepsilon)$		$f[x_1, x_1+\varepsilon, x_2] = \frac{f[x_1+\varepsilon, x_2] - f[x_1, x_1+\varepsilon]}{x_2 - x_1}$
		$f[x_1+\varepsilon, x_2]$	
x_2	$f[x_2] = f(x_2)$		$f[x_1+\varepsilon, x_2, x_2+\varepsilon] = \frac{f[x_2, x_2+\varepsilon] - f[x_1+\varepsilon, x_2]}{x_2 - x_1}$
		$f[x_2, x_2+\varepsilon]$	
$x_2 + \varepsilon$	$f[x_2+\varepsilon] = f(x_2+\varepsilon)$		

Passing to the limit as $\varepsilon \to 0$ gives us

x	$f(x)$	First diff.	Second diff.
x_0	$f[x_0] = f(x_0)$		
		$f'(x_0)$	
$x_0 + \varepsilon$	$f[x_0] = f(x_0)$		$f[x_0,x_0,x_1] = \dfrac{f[x_0,x_1] - f'(x_0)}{x_1 - x_0}$
		$f[x_0,x_1]$	
x_1	$f[x_1] = f(x_1)$		$f[x_0,x_1,x_1] = \dfrac{f'(x_1) - f[x_0,x_1]}{x_1 - x_0}$
		$f'(x_1)$	
$x_1 + \varepsilon$	$f[x_1] = f(x_1)$		$f[x_1,x_1,x_2] = \dfrac{f[x_1,x_2] - f'(x_1)}{x_2 - x_1}$
		$f[x_1,x_2]$	
x_2	$f[x_2] = f(x_2)$		$f[x_1,x_2,x_2] = \dfrac{f'(x_2) - f[x_1,x_2]}{x_2 - x_1}$
		$f'(x_2)$	
$x_2 + \varepsilon$	$f[x_2] = f(x_2)$		

Notice that once we get past the first differences there will no longer be a problem with division by 0, and we can form the remaining differences in the usual way.

The first few terms in the Newton formula then become

$$P(x) =$$

$$f[x_0] + f[x_0,x_0](x - x_0) + f[x_0,x_0,x_1](x - x_0)\left(x - x_0\right) + f[x_0,x_0,x_1,x_1](x - x_0)\left(x - x_0\right)(x - x_1)$$

$$= f(x_0) + f'(x_0)\ (x - x_0) + \frac{f[x_0,x_1] - f'(x_0)}{x_1 - x_0}(x - x_0)^2 + \cdots$$

I leave it as an exercise for you to confirm that this polynomial and its first derivative match $f(x)$ and its derivative at the interpolation points.