Neville's Method

Although the formula

$$L(x) = \sum_{k=0}^{n} y_k \, L_{n,k}(x)$$

that defines the Lagrange interpolating polynomial is correct, this is ultimately a clunky method for constructing and working with these interpolating polynomials. The problem with this method is that it requires a considerable amount of symbolic manipulation of polynomials. The basis functions $L_{n,k}(x)$ take a certain amount of work to construct, and forming the linear combination of the basis functions and simplifying the result down to a single polynomial all take a considerable amount of symbolic manipulation. Here is an alternative technique for computing and evaluating these polynomials which is much more useful in practice.

The technique requires some further special notation. Suppose we have n data points (x_k, y_k) that we would like to interpolate. Let us introduce the polynomials $p_{i,j}(x)$ with the following characteristics:

1. For any pair of indices i and j satisfying $0 \le i \le j \le n$, $p_{i,j}(x)$ is a polynomial that interpolates the points (x_i, y_i) through (x_j, y_j).
2. $p_{i,j}(x)$ is a polynomial of degree $j - i$.
3. $p_{i,i}(x) = y_i$ for all $0 \le i \le n$.
4. The $p_{i,j}(x)$ satisfy a *recurrence relation*

$$p_{i,j}(x) = \frac{(x - x_j)\, p_{i,j-1}(x) + (x_i - x)\, p_{i+1,j}(x)}{x_i - x_j}$$

To prove this recurrence relation we begin by noting that the expressions on either side of the equality are polynomials of degree $j - i$. To show that they are equal we simply have to show that they agree in value at $j - i + 1$ distinct values of x. The obvious candidates to check are the points x_k for $k = i$ through $k = j$.

At $x = x_i$ we have

$$\frac{(x_i - x_j)\, p_{i,j-1}(x_i) + (x_i - x_i)\, p_{i+1,j}(x_i)}{x_i - x_j} = \frac{(x_i - x_j)\, p_{i,j-1}(x_i)}{x_i - x_j}$$

$$= p_{i,j-1}(x_i) = y_i = p_{i,j}(x_i)$$

At $x = x_j$ we have

$$\frac{(x_j - x_j)\, p_{i,j-1}(x_j) + (x_i - x_j)\, p_{i+1,j}(x_j)}{x_i - x_j} = \frac{(x_i - x_j)\, p_{i+1,j}(x_j)}{x_i - x_j}$$

$$= p_{i+1,j}(x_j) = y_j = p_{i,j}(x_j)$$

For all other $x = x_k$ for k between i and j we have

$$\frac{(x_k - x_j)\, p_{i,j-1}(x_k) + (x_i - x_k)\, p_{i+1,j}(x_k)}{x_i - x_j} = \frac{(x_k - x_j)\, y_k + (x_i - x_k)\, y_k}{x_i - x_j}$$

$$= \frac{y_k\,(x_k - x_j + x_i - x_k)}{x_i - x_j} = y_k = p_{i,j}(x_k)$$

Once we have proved this recurrence relation we can use it as an alternative technique for constructing interpolating polynomials.

Perhaps the most useful application of this recurrence relation makes use of the fact that it allows us to evaluate interpolating polynomials at any x without actually having to construct the polynomial in question. This an improvement over the original definition of the Lagrange interpolating polynomials because the original method requires that we construct and then evaluate the basis functions to compute $L(x)$ for some x. This method allows us to evaluate $L(x)$ without actually having to construct any polynomials. This alternative method of computing $L(x)$ is known as *Neville's method*.

A further advantage of this method is that it easily allows us to do interpolation using only a subset i through j of the original list of data points.

Alternative Notation for Neville's Method

The notation $p_{i,j}(x)$ I used above is natural, but does not quite match the notation the author uses in the text. In preparation for later sections, the author uses a slightly different notation for his functions. The author defines the polynomial $Q_{i,j}(x)$ to be the polynomial that interpolates the points x_{i-j} through x_i. In other words,

$$Q_{i,j}(x) = p_{i-j,i}(x)$$

Using this notation the recursive formula for computing $Q_{i,j}(x)$ becomes

$$Q_{i,j}(x) = \frac{(x - x_{i-j})\, Q_{i,j-1}(x) - \left(x - x_i\right) Q_{i-1,j-1}(x)}{x_i - x_{i-j}}$$

This peculiar scheme for numbering the polynomials will make more sense when we get to the next section.