Definition

The point p is a *fixed point* of the function $g(x)$ if $g(p) = p$.

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The point p is a *root* of the function $f(x)$ if $f(x) = 0$.

Lemma

$f(x)$ has a root at p iff $g(x) = x - f(x)$ has a fixed point at p.

$g(x)$ has a fixed point at p iff $f(x) = x - g(x)$ has a root at p.

Observation

There is more than one way to convert a function that has a root at p into a function that has a fixed point at p.

Example

The function $f(x) = x^3 + 4x^2 - 10$ has a root somewhere in the interval [1,2]. Here are several functions that have a fixed point at that root.

$$g_1(x) = x - f(x) = x - x^3 - 4x^2 + 10$$

$$g_2(x) = \sqrt{\frac{10}{x} - 4x}$$

$$g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$$

$$g_4(x) = \sqrt{\frac{10}{4 + x}}$$

$$g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

Fixed point iteration

An interesting way to find a fixed point of a function $g(x)$ is the method of iteration.

1. Pick a point p_0 that you suspect is near the fixed point.

2. Compute $p_1 = g(p_0)$, $p_2 = g(p_1)$, …, $p_n = g(p_{n-1})$, …

3. If the sequence of p_n points converges, it converges to a fixed point of $g(x)$.

This iteration method will not always work for all functions $g(x)$ and all starting guesses p_0. The Jupyter notebook I have provided for chapter two shows some examples.

Why does iteration work?

Since there is more than one way to convert a root problem to a fixed point problem and some iterates appear to converge more quickly than others, it would be nice to understand why. The following theorem sheds some light on this.

Fixed Point Theorem

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$ for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$\left|g'(x)\right| \leq k \text{ for all } x \in (a,b)$$

Then, $g(x)$ has a unique fixed point p in $[a,b]$. Further, for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1})$$

converges to the unique fixed point p in $[a,b]$.

Proof We begin by showing that $g(x)$ has at least one fixed point in $[a,b]$. If $g(a) = a$ or $g(b) = b$, we are done. Otherwise, introduce

$$h(x) = g(x) - x$$

and note that $h(a) = g(a) - a > 0$ and $h(b) = g(b) - b < 0$ because $g(a)$ and $g(b)$ can not fall outside of $[a, b]$. By the intermediate value theorem, $h(x) = 0$ for some x in $[a,b]$. Note that a root of $h(x)$ is a fixed point of $g(x)$.

Next we show that $g(x)$ can not have more than one fixed point in $[a,b]$. Suppose by way of contradiction that p and q are both fixed points for $g(x)$ in $[a,b]$ with $p < q$. By the Mean Value Theorem we have that

$$\frac{g(p) - g(q)}{p - q} = g'(\xi)$$

for some ξ between p and q. Now note that

$$|p - q| = |g(p) - g(q)| = \left|g'(\xi)\right| |p - q| < |p - q|$$

which is a contradiction. Thus any fixed point in $[a,b]$ must be unique.

Now consider the sequence of iterated points p_n. Since g maps $[a,b]$ to $[a,b]$, there is no problem with the sequence wandering out of the interval. To show that it converges to p we use the fact that $|g'(x)| \le k$ for all $x \in (a,b)$ and the Mean Value Theorem to show

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi)||p_{n-1} - p| \le k\,|p_{n-1} - p|$$

Iterating this observation leads to

$$|p_n - p| \le k\,|p_{n-1} - p| \le k^2\,|p_{n-2} - p| \le \cdots \le k^n\,|p_0 - p|$$

and

$$\lim_{n\to\infty} |p_n - p| \le \lim_{n\to\infty} k^n\,|p_0 - p| = 0$$

Examples

We saw earlier that the functions

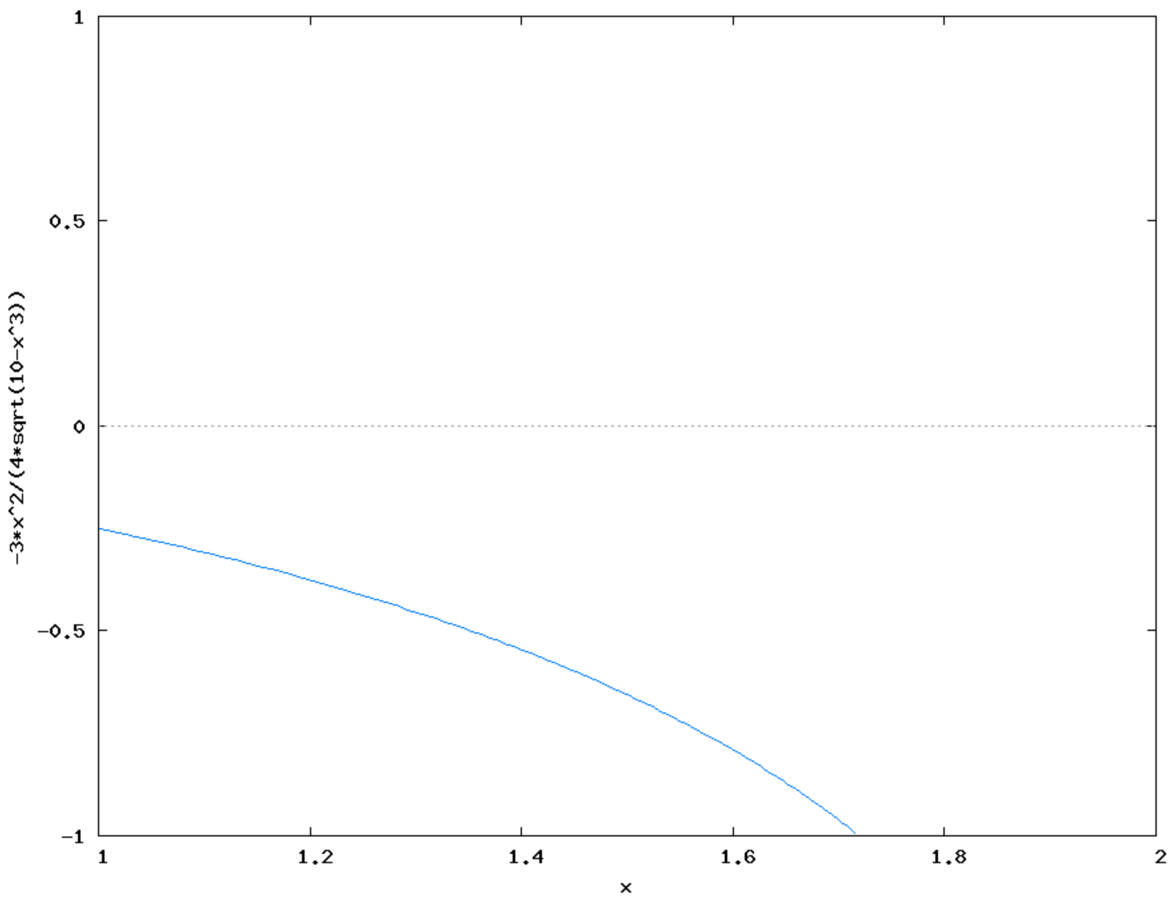
$$g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$$

$$g_4(x) = \sqrt{\frac{10}{4 + x}}$$

$$g_5(x) = x - \frac{x^3 + 4\,x^2 - 10}{3\,x^2 + 8\,x}$$

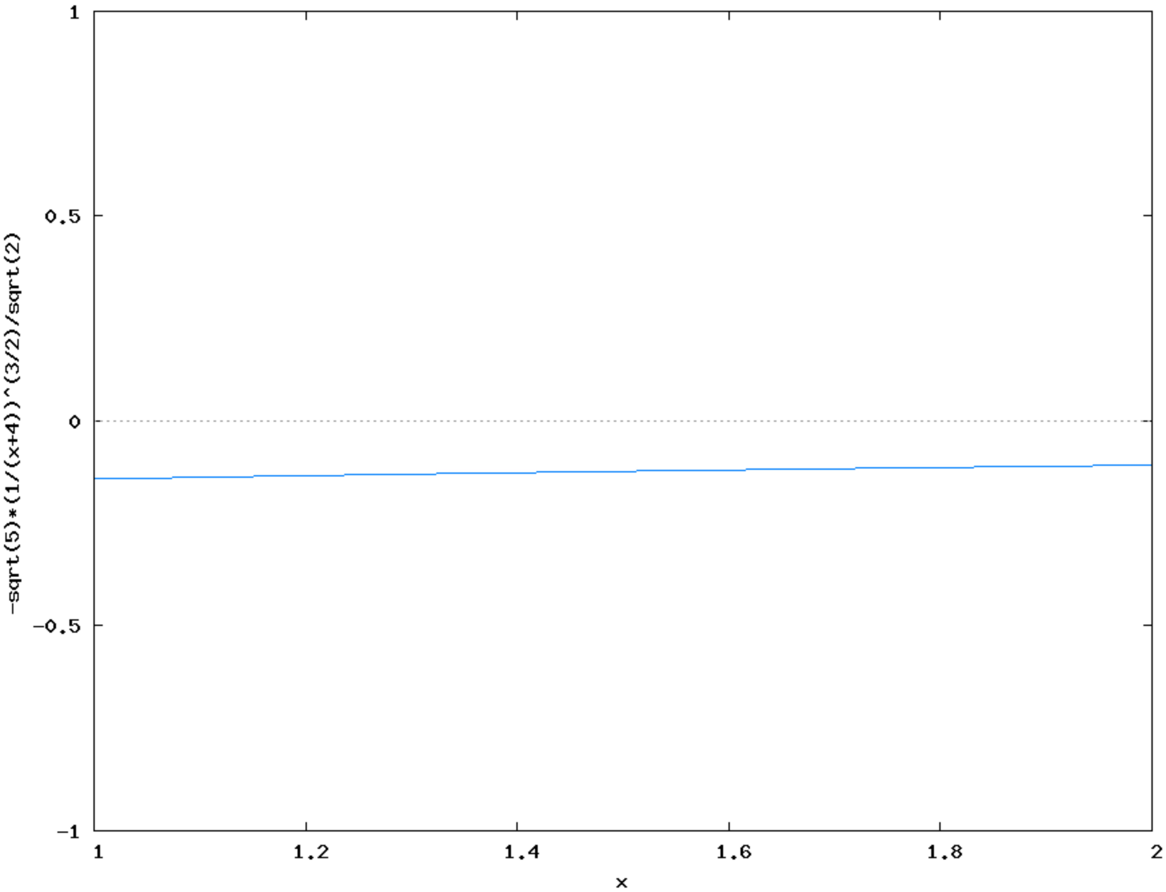
all had convergent iterates in the interval [1,2]. Looking at plots of their derivatives sheds some light on just why these converge and why the convergence gets better and better as we go down the list.

$$\left(\frac{1}{2}\sqrt{10 - x^3}\right)' = \frac{-\frac{3}{4}\,x^2}{\left(-x^3 + 10\right)^{\frac{1}{2}}}$$



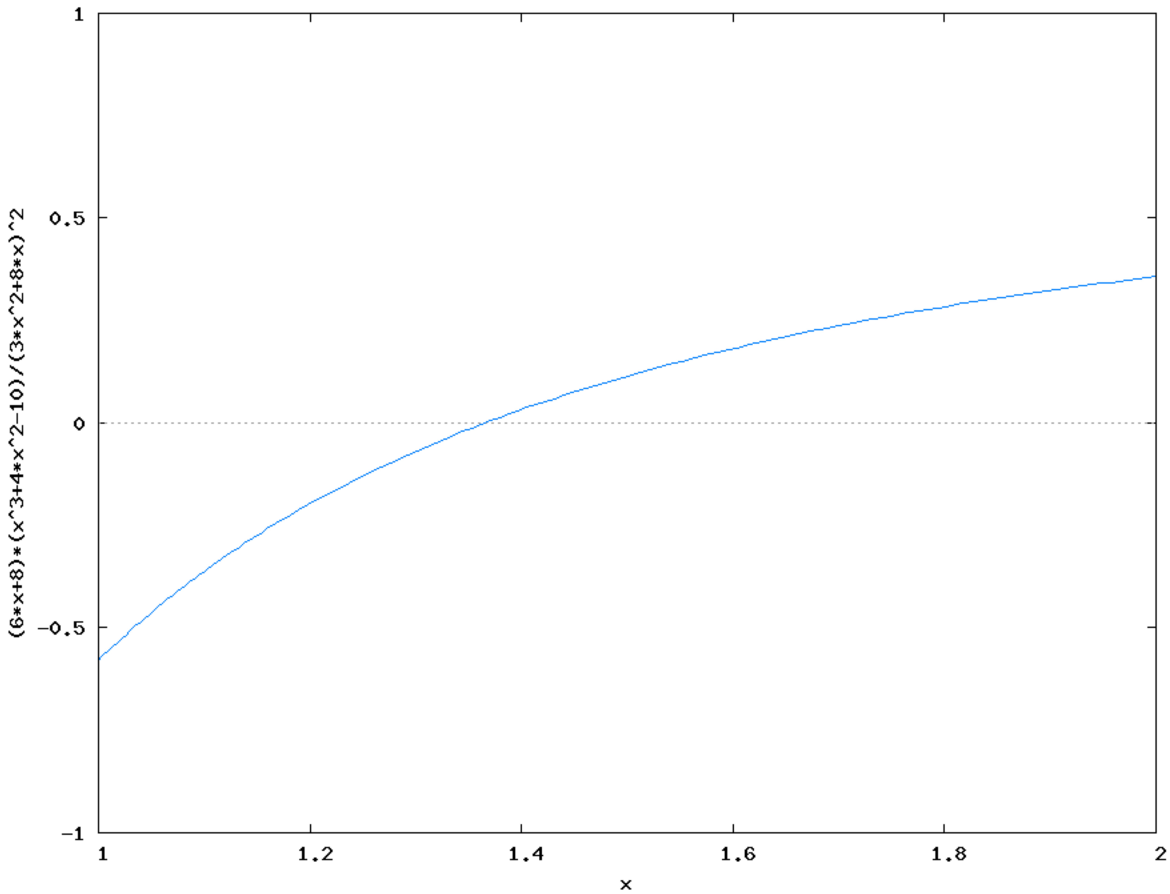
$$\frac{-\frac{3}{4}x^2}{\left(-x^3+10\right)^{\frac{1}{2}}}$$

$$\left(\sqrt{\frac{10}{4+x}}\right)' = -\left(\frac{5}{2}\right)^{\frac{1}{2}}\left(\frac{1}{x+4}\right)^{\frac{3}{2}}$$



$$-\left(\frac{5}{2}\right)^{\frac{1}{2}}\left(\frac{1}{x+4}\right)^{\frac{3}{2}}$$

$$\left(x-\frac{x^3+4\,x^2-10}{3\,x^2+8\,x}\right)'=\frac{(6\,x+8)\,\left(x^3+4\,x^2-10\right)}{\left(3\,x^2+8\,x\right)^2}$$



$$\frac{(6x+8)\left(x^3+4x^2-10\right)}{\left(3x^2+8x\right)^2}$$

Bounding the convergence

In cases where the fixed point theorem predicts that the sequence of iterates will converge to a fixed point, it might also be useful to estimate how quickly the convergence will happen. More specifically, we will often want to know how many iterations are necessary to guarantee that the n^{th} iterate p_n lands within some distance ε of the fixed point p. To help with this the text offers the following

Corollary If $g(x)$ satisfies the conditions of the fixed point theorem on an interval $[a,b]$ and $|g'(x)| \leq k$ on that interval then

$$|p_n - p| \leq \frac{k^n}{1-k} |p_1 - p_0| \text{ for all } n \geq 1$$

Proof Using the same reasoning we used in the proof of the fixed point theorem we have

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \leq k\,|p_n - p_{n-1}| \leq \cdots \leq k^n\,|p_1 - p_0|$$

We also have

$$|p_{n+m} - p_n| = |p_{n+m} - p_{n+m-1} + p_{n+m-1} - p_{n+m-2} + \cdots + p_{n+1} - p_n|$$

$$\leq |p_{n+m} - p_{n+m-1}| + |p_{n+m-1} - p_{n+m-2}| + \cdots + |p_{n+1} - p_n|$$

$$\leq \left(1 + k^1 + k^2 + \cdots + k^{m-1}\right) k^n \; |p_1 - p_0|$$

Taking the limit as $m \to \infty$ the sum of the powers of k in the last expression converges to the geometric series, which sums to $1/(1-k)$, while $p_{n+m} \to p$.