The finite difference method

Consider a linear boundary value problem.

$$y''(x) = p(x)\, y'(x) + q(x)\, y(x) + r(x)$$

$$y(a) = \alpha$$

$$y(b) = \beta$$

In the method of finite differences we seek an approximate solution to this boundary value problem by setting up a grid of $N + 2$ equally spaced points x_i with $x_0 = a$ and $x_{N+1} = b$:

$$h = \frac{b-a}{N+1}$$

$$x_i = a + i\, h$$

The method seeks to compute estimates for $y(x_i)$ at each of the interior points x_i for i ranging from 1 to N by replacing the derivative terms with finite difference estimates and solving a set of equations. Specifically, we replace the term $y''(x_i)$ with an $O(h^2)$ centered difference formula

$$y''(x_i) = \frac{y(x_{i+1}) - 2\, y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12}\, y^{(4)}(\xi_i)$$

and we replace the term $y'(x_i)$ with an $O(h^2)$ centered difference formula

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2\, h} - \frac{h^2}{6}\, y^{(3)}(\eta_i)$$

Making these substitutions gives

$$\frac{y(x_{i+1}) - 2\, y(x_i) + y(x_{i-1})}{h^2} = p(x_i) \left(\frac{y(x_{i+1}) - y(x_{i-1})}{2\, h}\right) + q(x_i)\, y(x_i) + r(x_i)$$

for each i ranging from 1 to N up to $O(h^2)$. If we let w_i be the solution of this equation for $y(x_i)$ for each of these i values with $w_0 = \alpha$ and $w_{N+1} = \beta$, we get a coupled system of linear equations in w_1 through w_N:

$$\frac{w_{i+1} - 2\, w_i + w_{i-1}}{h^2} = p(x_i) \left(\frac{w_{i+1} - w_{i-1}}{2\, h}\right) + q(x_i)\, w_i + r(x_i)$$

Rearranging this equation slightly produces

$$-\left(1 + \frac{h}{2}\, p(x_i)\right) w_{i-1} + \left(2 + h^2\, q(x_i)\right) w_i - \left(1 - \frac{h}{2}\, p(x_i)\right) w_{i+1} = -h^2\, r(x_i)$$

Solving the system of equations

Because the original boundary value problem was linear, the set of N equations in the N unknowns w_1 through w_N

is linear and can be written in matrix form.

$$\begin{bmatrix} 2+h^2\,q(x_1) & -\left(1-\frac{h}{2}\,p(x_1)\right) & 0 & \cdots & 0 & 0 \\ -\left(1+\frac{h}{2}\,p(x_2)\right) & 2+h^2\,q(x_2) & -\left(1-\frac{h}{2}\,p(x_2)\right) & \cdots & 0 & 0 \\ 0 & -\left(1+\frac{h}{2}\,p(x_3)\right) & 2+h^2\,q(x_3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 2+h^2\,q(x_{N-1}) & -\left(1-\frac{h}{2}\,p(x_{N-1})\right) \\ 0 & 0 & 0 & \cdots & -\left(1+\frac{h}{2}\,p(x_N)\right) & 2+h^2\,q(x_N) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_{N-1} \\ w_N \end{bmatrix}$$

$$= \begin{bmatrix} -h^2\,r(x_1) + \left(1+\frac{h}{2}\,p(x_1)\right)\alpha \\ -h^2\,r(x_2) \\ -h^2\,r(x_3) \\ \vdots \\ -h^2\,r(x_{N-1}) \\ -h^2\,r(x_N) + \left(1-\frac{h}{2}\,p(x_N)\right)\beta \end{bmatrix}$$

or

$$A\,\mathbf{w} = \mathbf{b}$$

The coefficient matrix is tridiagonal, so we can solve this system most directly by Crout factorization of the matrix A.

$$A = LU$$

To solve the system we solve the two problems

$$L\,\mathbf{z} = \mathbf{b}$$

$$U\,\mathbf{w} = \mathbf{z}$$

by back substitution.

Extrapolation

Because the w_i we solved for above were obtained by approximating solutions $y(x_i)$ to the original boundary value problem by an $O(h^2)$ process, we have that

$$y(x_i) = w_i(h) + O(h^2)$$

Here we have made the dependence of w_i on h explicit. Because $y(x_i)$ does not depend on h, this equation is a candidate for the Richardson extrapolation:

$$y(x_i) = w_i(h) + k_1\, h^2 + k_2\, h^4 + \cdots$$

$$y(x_i) = w_i(h/2) + \frac{k_1}{4}\, h^2 + \frac{k_2}{16}\, h^4 + \cdots$$

Combining these equations to eliminate the $O(h^2)$ terms gives

$$y(x_i) = \frac{4\, w_i(h/2) - w_i(h)}{3} + O(h^4)$$

To apply this formula, we compute a first set of w_i values from a step size h and a second set of w_i from a step size $h/2$ and throw away every other value from the second set. We then apply the formula above to the remaining points to get an $O(h^4)$ approximation for the corresponding $y(x_i)$ values.

To get even more accurate results we could apply an extrapolation to the extrapolation. See the textbook for details.