Inverting a Differential Operator

Consider the BVP

$$-\frac{d}{dx}\left(k(x)\,\frac{du}{dx}\right)=f(x)$$

$$u(a)=\frac{du}{dx}(b)=0$$

It is possible to show that the differential operator

$$L\,u=-\frac{d}{dx}\left(k(x)\,\frac{du}{dx}\right)$$

is a symmetric operator on the space $C_m^{2}[a,b]$ of twice continuously differentiable functions that satisfy mixed boundary conditions $u(a)=\frac{d\,u}{d\,x}(b)=0$ on the interval $[a,b]$.

Our goal is to develop a linear operator M that maps the space of functions $C[a,b]$ to the space $C_m^{2}[a,b]$ in such a way that $M\,f=u$ whenever $L\,u=f$. M is the *inverse operator* to L in the sense that

$$L\,M\,f=L\,u=f$$

and

$$M\,L\,u=M\,f=u$$

We will see that in order to construct the inverse of the differential operator L we will have to use an *integral operator* that takes the form

$$u=M\,f=\int_a^b G(x;\,y)\,f(y)\,d\,y$$

The function $G(x\,;\,y)$ is called the *kernel* of the linear operator M, or the *Green's function* for the linear differential operator L.

A Simple Example

To demonstrate that it is indeed possible to invert a differential operator into an integral operator, we start with a simple example. The differential operator

$$L\,u=-\frac{d}{dx}\left(k(x)\,\frac{du}{dx}\right)$$

is directly invertible by integration:

$$-\frac{d}{dx}\left(k(x)\,\frac{du}{dx}\right)=f(x)$$

we integrate both sides with respect to x from x to b:

$$\int_x^b f(y)\,\mathrm{d}y = \int_x^b -\frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = \int_b^x \frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = k(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}(x) - k(b)\,\frac{\mathrm{d}u}{\mathrm{d}x}(b) = k(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}(x)$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}(x) = \frac{1}{k(x)}\int_x^b f(y)\,\mathrm{d}y$$

Integrating a second time on both sides from a to x gives

$$\int_a^x \frac{\mathrm{d}u}{\mathrm{d}x}(z)\,\mathrm{d}z = u(x) - u(a) = u(x) = \int_a^x \frac{1}{k(z)}\int_z^b f(y)\,\mathrm{d}y\,\mathrm{d}z$$

By interchanging the order of the z and y integrals on the right we get

$$u(x) = \int_a^b \left(\int_a^{min(x,y)} \frac{1}{k(z)}\,\mathrm{d}z\right) f(y)\,\mathrm{d}y$$

We now see that the expression

$$G(x\,;\,y) = \int_a^{min(x,y)} \frac{1}{k(z)}\,\mathrm{d}z$$

allows us to cast this solution in the form of an integral operator:

$$u(x) = M\,f = \int_a^b G(x\,;\,y)\,f(y)\,\mathrm{d}y$$

One final comment about this example. The one thing that allowed the problem to slide through as readily as it did was the special boundary condition $\frac{\mathrm{d}u}{\mathrm{d}x}(b) = 0$. In fact, if we tried to impose the more conventional boundary condition of $u(b) = 0$ we would find that the mathematics of computing the Green's function becomes much more difficult.

A More Ambitious Example

Next we turn to a more challenging BVP

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}\right) + R(x)\,u = F(x)$$

$$u(a) = u(b) = 0$$

This is such a challenging problem that we have to assume that we are provided with a little help. Suppose we have already solved the homogeneous form of the this problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}\right) + R(x)\,u = 0$$

and have obtained two linearly independent solutions $v_1(x)$ and $v_2(x)$. To use the solutions of the homogeneous problem to help us solve the nonhomogeneous problem, we deploy the method of variation of parameters. This method assumes that the solution to the nonhomogeneous problem takes the form

$$u(x) = c_1(x)\, v_1(x) + c_2(x)\, v_2(x)$$

We proceed by differentiating this function with respect to x and then substituting into the equation. The first step is to compute the derivative of u(x) with respect to x:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = c_1(x)\,\frac{\mathrm{d}v_1}{\mathrm{d}x} + c_2(x)\,\frac{\mathrm{d}v_2}{\mathrm{d}x} + \frac{\mathrm{d}c_1}{\mathrm{d}x}\,v_1(x) + \frac{\mathrm{d}c_2}{\mathrm{d}x}\,v_2(x)$$

Because this expression will lead to a mess, we exercise our ability to place constraints on the coefficient functions $c_1(x)$ and $c_1(x)$ by demanding that part of this expression vanish:

$$\frac{\mathrm{d}c_1}{\mathrm{d}x}\,v_1(x) + \frac{\mathrm{d}c_2}{\mathrm{d}x}\,v_2(x) = 0$$

This immediately causes the expression for du/dx to simplify to

$$\frac{\mathrm{d}u}{\mathrm{d}x} = c_1(x)\,\frac{\mathrm{d}v_1}{\mathrm{d}x} + c_2(x)\,\frac{\mathrm{d}v_2}{\mathrm{d}x}$$

Substituting this simplified expression into the ODE gives

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\left(c_1(x)\,\frac{\mathrm{d}v_1}{\mathrm{d}x} + c_2(x)\,\frac{\mathrm{d}v_2}{\mathrm{d}x}\right)\right) + R(x)\left(c_1(x)\,v_1(x) + c_2(x)\,v_2(x)\right) = F(x)$$

differentiating on the left gives us

$$-\frac{\mathrm{d}P}{\mathrm{d}x}\left(c_1(x)\,\frac{\mathrm{d}v_1}{\mathrm{d}x} + c_2(x)\,\frac{\mathrm{d}v_2}{\mathrm{d}x}\right) - P(x)\left(c_1(x)\,\frac{\mathrm{d}^2v_1}{\mathrm{d}x^2} + c_2(x)\,\frac{\mathrm{d}^2v_2}{\mathrm{d}x^2} + \frac{\mathrm{d}c_1}{\mathrm{d}x}\,\frac{\mathrm{d}v_1}{\mathrm{d}x} + \frac{\mathrm{d}c_1}{\mathrm{d}x}\,\frac{\mathrm{d}v_2}{\mathrm{d}x}\right)$$

$$+ R(x)\left(c_1(x)\,v_1(x) + c_2(x)\,v_2(x)\right) = F(x)$$

After reorganizing the terms on the left, we get something like this:

$$\left(-c_1(x)\,\frac{\mathrm{d}P}{\mathrm{d}x}\,\frac{\mathrm{d}v_1}{\mathrm{d}x} - c_1(x)\,P(x)\,\frac{\mathrm{d}^2v_1}{\mathrm{d}x^2} + c_1(x)\,R(x)\,v_1(x)\right) +$$

$$\left(-c_2(x)\,\frac{\mathrm{d}P}{\mathrm{d}x}\,\frac{\mathrm{d}v_2}{\mathrm{d}x} - c_2(x)\,P(x)\,\frac{\mathrm{d}^2v_2}{\mathrm{d}x^2} + c_2(x)\,R(x)\,v_2(x)\right) -$$

$$P(x)\left(\frac{\mathrm{d}c_1}{\mathrm{d}x}\,\frac{\mathrm{d}v_1}{\mathrm{d}x} + \frac{\mathrm{d}c_1}{\mathrm{d}x}\,\frac{\mathrm{d}v_2}{\mathrm{d}x}\right) = F(x)$$

or

$$c_1(x)\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\,\frac{\mathrm{d}v_1}{\mathrm{d}x}\right) + R(x)\,v_1(x)\right) + c_2(x)\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\,\frac{\mathrm{d}v_2}{\mathrm{d}x}\right) + R(x)\,v_2(x)\right) -$$

$$P(x)\left(\frac{\mathrm{d}c_1}{\mathrm{d}x}\,\frac{\mathrm{d}v_1}{\mathrm{d}x} + \frac{\mathrm{d}c_1}{\mathrm{d}x}\,\frac{\mathrm{d}v_2}{\mathrm{d}x}\right) = F(x)$$

Because the functions $v_1(x)$ and $v_2(x)$ are both solutions of the homogeneous problem, the first two sets of terms vanish and we are left with

$$-P(x)\left(\frac{dc_1}{dx}\frac{dv_1}{dx}+\frac{dc_1}{dx}\frac{dv_2}{dx}\right)=F(x)$$

We now have two equations to solve, the condition we imposed earlier, and this new equation:

$$\frac{dc_1}{dx}\,v_1(x)+\frac{dc_2}{dx}\,v_2(x)=0$$

$$-P(x)\left(\frac{dc_1}{dx}\frac{dv_1}{dx}+\frac{dc_2}{dx}\frac{dv_2}{dx}\right)=F(x)$$

Close examination of this pair of equations shows that this can be written as a system of equations.

$$\begin{bmatrix} v_1(x) & v_2(x) \\ \frac{dv_1}{dx} & \frac{dv_2}{dx} \end{bmatrix}\begin{bmatrix} \frac{dc_1}{dx} \\ \frac{dc_2}{dx} \end{bmatrix}=\begin{bmatrix} 0 \\ -F(x)/P(x) \end{bmatrix}$$

This system can be solved by multiplying both sides by the inverse of the matrix on the left.

$$\begin{bmatrix} \frac{dc_1}{dx} \\ \frac{dc_2}{dx} \end{bmatrix}=\begin{bmatrix} v_1(x) & v_2(x) \\ \frac{dv_1}{dx} & \frac{dv_2}{dx} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -F(x)/P(x) \end{bmatrix}$$

$$=\frac{1}{\frac{dv_2}{dx}\,v_1(x)-\frac{dv_1}{dx}\,v_2(x)}\begin{bmatrix} \frac{dv_2}{dx} & -v_2(x) \\ -\frac{dv_1}{dx} & v_1(x) \end{bmatrix}\begin{bmatrix} 0 \\ -F(x)/P(x) \end{bmatrix}$$

The quantity

$$W(x)=\frac{dv_2}{dx}\,v_1(x)-\frac{dv_1}{dx}\,v_2(x)$$

is known as the *Wronskian* of the functions $v_1(x)$ and $v_2(x)$. The Wronskian has a number of special properties. One of these properties is that if $v_1(x)$ and $v_2(x)$ are linearly independent functions on $[a,b]$ then $W(x)$ for all x in $[a,b]$. Another useful property is that if $v_1(x)$ and $v_2(x)$ are solutions of the homogeneous problem then the function $W(x)$ $P(x)$ is actually a constant over the entire interval. We can see this by computing its derivative with respect to x:

$$\frac{d}{dx}\left(W(x)\,P(x)\right)=W'(x)\,P(x)+W(x)\,P'(x)$$

$$P(x)\,\frac{d^2v_2}{dx^2}\,v_1(x)-P(x)\,\frac{d^2v_1}{dx^2}\,v_2(x)+P(x)\,\frac{dv_2}{dx}\frac{dv_1}{dx}-P(x)\,\frac{dv_1}{dx}\frac{dv_2}{dx}+P'(x)\frac{dv_2}{dx}\,v_1(x)-P'(x)\frac{dv_1}{dx}\,v_2(x)$$

$$= v_1(x)\left(P(x)\,\frac{d^2v_2}{dx^2} + P'(x)\frac{dv_2}{dx}\right) - v_2(x)\left(P(x)\,\frac{d^2v_1}{dx^2} + P'(x)\frac{dv_1}{dx}\right) + \left(P(x)\,\frac{dv_2}{dx}\,\frac{dv_1}{dx} - P(x)\,\frac{dv_1}{dx}\,\frac{dv_2}{dx}\right)$$

The last of these terms vanishes. As for the other two terms, adding and subtracting some terms $R(x)\ v_1(x)\ v_2(x)$ gives

$$= v_1(x)\left(P(x)\,\frac{d^2v_2}{dx^2} + P'(x)\frac{dv_2}{dx} - R(x)\,v_2(x)\right) - v_2(x)\left(P(x)\,\frac{d^2v_1}{dx^2} + P'(x)\frac{dv_1}{dx} - R(x)\,v_1(x)\right) + 0$$

$$= 0 + 0 + 0$$

Thus, the quantity $W(x)\ P(x)$ is a constant over the entire interval, so we can replace it with $1/k$ where it appears in the matrix equation we derive earlier. We now have

$$\begin{bmatrix} \frac{dc_1}{dx} \\ \frac{dc_2}{dx} \end{bmatrix} = \begin{bmatrix} v_1(x) & v_2(x) \\ \frac{dv_1}{dx} & \frac{dv_2}{dx} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -F(x)/P(x) \end{bmatrix}$$

$$= \frac{1}{\frac{dv_2}{dx}\,v_1(x) - \frac{dv_1}{dx}\,v_2(x)} \begin{bmatrix} \frac{dv_2}{dx} & -v_2(x) \\ -\frac{dv_1}{dx} & v_1(x) \end{bmatrix} \begin{bmatrix} 0 \\ -F(x)/P(x) \end{bmatrix}$$

$$= \frac{1}{W(x)} \begin{bmatrix} \frac{dv_2}{dx} & -v_2(x) \\ -\frac{dv_1}{dx} & v_1(x) \end{bmatrix} \begin{bmatrix} 0 \\ -F(x)/P(x) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{dv_2}{dx} & -v_2(x) \\ -\frac{dv_1}{dx} & v_1(x) \end{bmatrix} \begin{bmatrix} 0 \\ -F(x)/(W(x)P(x)) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{dv_2}{dx} & -v_2(x) \\ -\frac{dv_1}{dx} & v_1(x) \end{bmatrix} \begin{bmatrix} 0 \\ -kF(x) \end{bmatrix}$$

or

$$\begin{bmatrix} \frac{dc_1}{dx} \\ \frac{dc_2}{dx} \end{bmatrix} = \begin{bmatrix} k\,v_2(x)\,F(x) \\ -\,k\,v_1(x)\,F(x) \end{bmatrix}$$

This now leads to expressions for $c_1(x)$ and $c_2(x)$:

$$c_1(x) = \int_a^x k\, v_2(x)\, F(x)\, \mathrm{d}\,x + a_1$$

$$c_2(x) = \int_x^b k\, v_1(x)\, F(x)\, \mathrm{d}\,x + a_2$$

Notice the trick we played with the second integral to effectively absorb the minus sign.

The last step is to bring in the boundary conditions

$$u(a) = u(b) = 0$$

to determine what the unknown constants a_1 and a_2 are. Applying the first boundary condition yields

$$0 = u(a) = c_1(a)\, v_1(a) + c_2(a)\, v_2(a)$$

$$= \left(\int_a^a k\, v_2(x)\, F(x)\, \mathrm{d}\,x + a_1\right) v_1(a) + \left(\int_a^b k\, v_1(x)\, F(x)\, \mathrm{d}\,x + a_2\right) v_2(a)$$

$$= a_1\, v_1(a) + \left(\int_a^b k\, v_1(x)\, F(x)\, \mathrm{d}\,x + a_2\right) v_2(a)$$

We can get this to equal 0 by imposing two conditions:

$$v_1(a) = 0$$

$$a_2 = -\int_a^b k\, v_1(x)\, F(x)\, \mathrm{d}\,x$$

Similarly,

$$0 = u(b) = c_1(b)\, v_1(b) + c_2(b)\, v_2(b)$$

$$= \left(\int_a^b k\, v_2(x)\, F(x)\, \mathrm{d}\,x + a_1\right) v_1(b) + \left(\int_b^b k\, v_1(x)\, F(x)\, \mathrm{d}\,x + a_2\right) v_2(b)$$

$$= \left(\int_a^b k\, v_2(x)\, F(x)\, \mathrm{d}\,x + a_1\right) v_1(b) + a_2\, v_2(b)$$

This leads us to impose conditions

$$v_2(b) = 0$$

$$a_1 = -\int_a^b k\, v_2(x)\, F(x)\, \mathrm{d}\,x$$

An obvious question here is whether we are in fact free to impose the conditions

$$v_1(a) = 0$$

$$v_2(b) = 0$$

We can get away with this, because at the start of the process we simply demanded that $v_1(x)$ and $v_2(x)$ be solutions to the homogeneous problem and that they be linearly independent. A little reflection will make it clear that we are free to impose these additional conditions without violating independence.

We now go on to write out the full solution.

$$u(x) = c_1(x)\, v_1(x) + c_2(x)\, v_2(x)$$

$$= \left(\int_a^x k\, v_2(y)\, F(y)\, \mathrm{d}\, y - \int_a^b k\, v_2(y)\, F(y)\, \mathrm{d}\, y\right) v_1(x) + \left(\int_x^b k\, v_1(y)\, F(y)\, \mathrm{d}\, x - \int_a^b k\, v_1(y)\, F(y)\, \mathrm{d}\, y\right) v_2(x)$$

$$= \left(-\int_x^b k\, v_2(y)\, F(y)\, \mathrm{d}\, y\right) v_1(x) + \left(-\int_a^x k\, v_1(y)\, F(y)\, \mathrm{d}\, y\right) v_2(x)$$

If we define

$$G(x\,;\,y) = \begin{cases} -k\, v_1(y)\, v_2(x) & y \le x \\ -k\, v_2(y)\, v_1(x) & y > x \end{cases}$$

this takes the desired form:

$$u(x) = \int_a^b G(x\,;\,y)\, F(y)\, \mathrm{d}\, y$$

Interpreting the Green's Function

We have just seen that in some cases it is possible to invert a differential operator L and solve an ODE

$$L\, u = f$$

by expressing the solution via an integral operator.

$$u = L^{-1} f = \int_a^b G(x\,;\,y)\, f(y)\, \mathrm{d}\, y$$

An important observation about this integral operator is that it is a linear operator. For example, suppose that it were possible to write the function f(x) as a combination of two functions with separate support.

$$f(x) = f_1(x) + f_2(x)$$

By linearity of the integral operator, we could then write the solution as a combination of two pieces.

$$u(x) = u_1(x) + u_2(x) = \int_a^b G(x\,;\,y)\, f_1(y)\, \mathrm{d}\, y + \int_a^b G(x\,;\,y)\, f_2(y)\, \mathrm{d}\, y$$

In this situation we say that the term

$$\int_a^b G(x\,;\,y)\, f_1(y)\, \mathrm{d}\, y$$

models the influence of $f_1(x)$ on the solution, while

$$\int_a^b G(x\,;\,y)\,f_2(y)\,\mathrm{d}\,y$$

models the influence of $f_2(x)$ on the solution.

Now consider an obvious generalization of this idea. Suppose we introduce a function

$$d_\Delta(x) = \begin{cases} \dfrac{x+\Delta x}{(\Delta x)^2} & -\Delta x < x < 0 \\ -\dfrac{x-\Delta x}{(\Delta x)^2} & 0 \le x < \Delta x \\ 0 & otherwise \end{cases}$$

This is a "spike" function with a base of width 2 Δx and height $1/\Delta x$. We can translate this spike to different locations by forming translates $d_\Delta(x - \xi)$.

These spike functions are reminiscent of the basis functions we used for the finite element method. In particular, we can use spike functions to construct approximations for smooth functions.

$$f(x) \approx \sum_{i=1}^{n} f_i\, d_\Delta(x - \xi_i)$$

where the points ξ_i for $i = 1$ to n are some set of evenly spaced sample points on the interval $[a,b]$. Writing the forcing function $f(x)$ as a linear combination of spike functions allows us to approximate the solution to $Lu = f$ by

$$u(x) \approx \sum_{i=1}^{n} f_i \left(\int_a^b G(x\,;\,y)\, d_\Delta(y - \xi_i)\,\mathrm{d}\,y \right)$$

What can we say about the term

$$\int_a^b G(x\,;\,y)\, d_\Delta(y - \xi_i)\,\mathrm{d}\,y$$

that appears here? Since the spike function has a very narrow support centered on the point $y = \xi_i$, what we appear to be computing here is a weighted average of the function $G(x\,;\,y)$ in the immediate neighborhood of $y = \xi_i$. In the limit as $\Delta y \to 0$, we should expect this weighted average to converge to the value of $G(x\,;\,y)$ at $y = \xi_i$, $G(x\,;\,\xi_i)$.

$$G(x\,;\,\xi) = \lim_{\Delta \to 0} \int_a^b G(x\,;\,y)\, d_\Delta(y - \xi)\,\mathrm{d}\,y$$

Further properties of $d_\Delta(x)$

We can further formalize some of the ideas from the previous section by making some observations about the behavior of the function $d_\Delta(x - \xi)$ and its limit as $\Delta x \to 0$, $\delta(x-\xi)$.

$$\int_a^b d_\Delta(x - \xi)\,\mathrm{d}\,x = 1$$

$$\int_a^b \delta(x - \xi)\,\mathrm{d}\,x = 1$$

$$\int_a^b g(x)\; d_\Delta(x - \xi)\,\mathrm{d}\,x \approx g(\xi)$$

$$\int_a^b g(x)\;\delta(x - \xi)\,\mathrm{d}\,x = g(\xi)$$

This last property of $\delta(x\text{-}\xi)$ is the most interesting for our purposes:

$$\int_a^b G(x\,;\,y)\;\delta(y - \xi)\,\mathrm{d}\,y = G(x\,;\,\xi)$$

This observation leads to the following important interpretation of the Green's function:

$G(x\,;\,y)$ is the response of the system to a forcing function $f(x) = \delta(x\text{-}y)$

The significance of this observation is that in many cases we can predict what the response of the system we are modeling will be to such an impulse without having to explicitly solve the differential equation. That alternative reasoning will lead us directly to an expression for the Green's function, and once we have that Green's function we can write the solution to $L\,u = f$ directly as

$$u(x) = \int_a^b G(x\,;\,y)\,f(y)\,\mathrm{d}\,y$$

Computing Green's functions directly

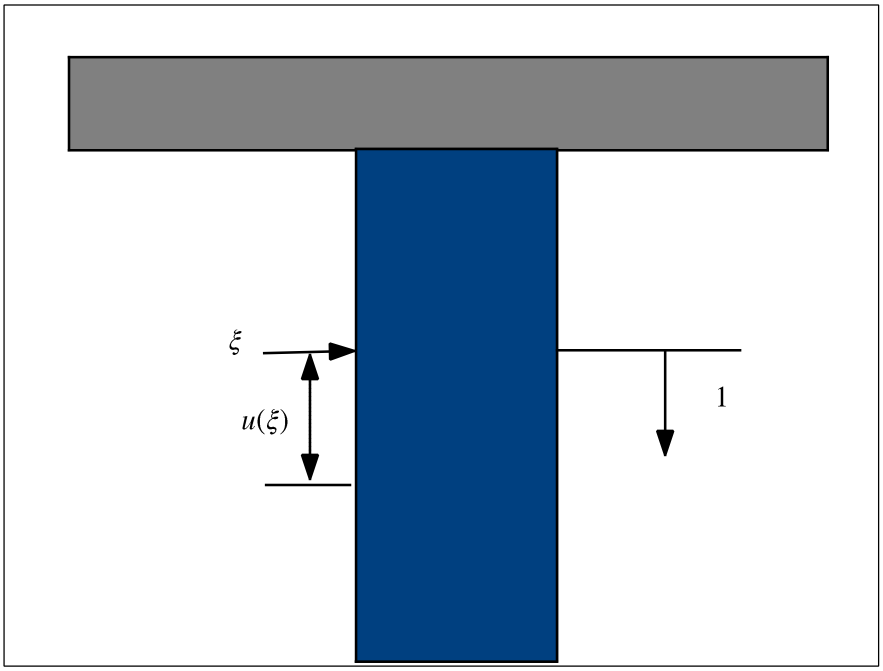
To see an example of how we can compute a Green's function directly, let us return to the problem of the hanging bar. In the simplest version of this problem, the function $k(x)$ reduces to a constant k and we have to solve the problem

$$-k\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = f(x)$$

$$u(0) = 0$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}(l) = 0$$

The Green's function method asks us to predict the response of this system to a forcing function that takes the form $f(x) = \delta(x - \xi)$, where ξ is some point between 0 and l.



The physical interpretation of this situation is very simple. We are applying one unit of force concentrated over an infinitely thin range of contact centered at $x = \xi$. Since the bar has a uniform Hooke's constant of k, we can easily compute how the bar will respond to this situation. The portion of the bar between 0 and ξ will stretch uniformly in accordance with the underlying physical principle

$$stress = k\ strain$$

or

$$1 = k\,\frac{u(\xi)}{\xi}$$

or

$$u(\xi) = \frac{\xi}{k}$$

At points x with $0 \le x \le \xi$, the bar will experience an amount of stretching which is proportional to the ratio x/ξ:

$$u(x) = \frac{x}{k}$$

At points x with $\xi \le x \le l$ the bar will experience no stretching, only displacement:

$$u(x) = \frac{\xi}{k}$$

We can summarize this by saying that the response of this system to a forcing function $\delta(x - \xi)$ is

$$G(x\,;\,\xi) = \begin{cases} x/k & x < \xi \\ \xi/k & x \geq \xi \end{cases}$$

We can then use this Green's function to compute the response of this system to any forcing function $f(x)$:

$$u(x) = \int_0^l G(x\,;\,y)\, f(y)\,\mathrm{d}\,y = \int_0^l f(y) \begin{cases} x/k & y > x \\ y/k & y \leq x \end{cases} \mathrm{d}\,y = \int_0^x y/k\, f(y)\,\mathrm{d}\,y + \int_x^l x/k\, f(y)\,\mathrm{d}\,y$$

Note that we can also express this Green's function as an integral:

$$G(x\,;\,y) = \int_0^{min(x,y)} \frac{1}{k}\,\mathrm{d}\,y$$

In this form it matches the Green's function we computed at the start of these notes.

A second example

Consider now a slight variant of the problem above.

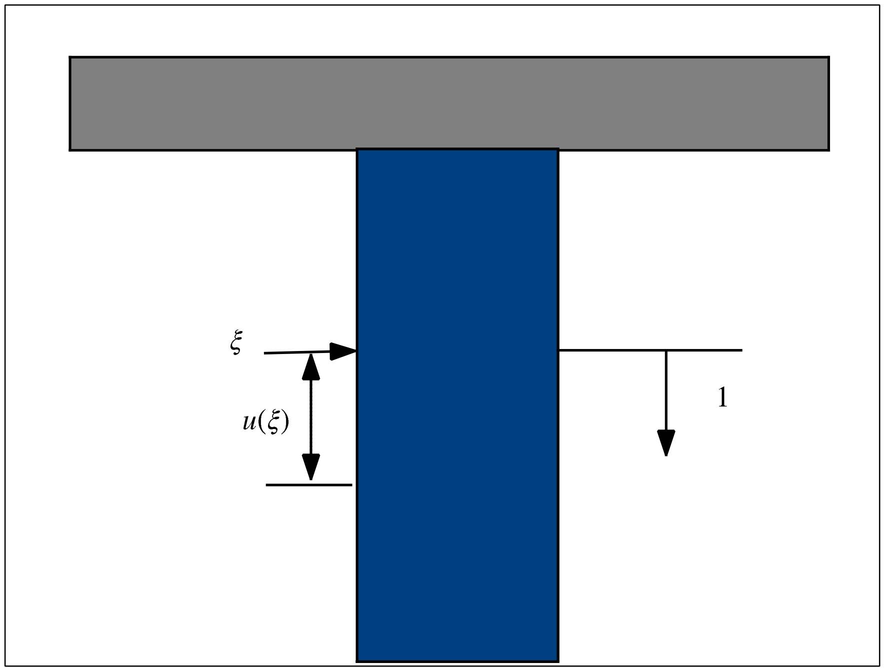
$$-k\,\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x)$$

$$u(0) = 0$$

$$u(l) = 0$$

In this version of the problem, the bar is fixed at both ends instead of being fixed only at the top and free to move at the bottom.

If we replay the physics of the situation above



with the modification that the bottom end is now fixed, we see that once again the portion of the bar between $x = 0$ and $x = \xi$ will experience stretching, while the portion of the bar between $x = \xi$ and $x = l$ is compressed. This means that bar will "push back" against the external force with two force terms: one is the stress on the upper part of the bar which is being stretched and the other is the stress on the lower part of the bar which is being compressed.

$$1 = k \frac{u(\xi)}{\xi} + k \frac{u(\xi)}{l - \xi}$$

Solving this for $u(\xi)$ produces

$$u(\xi) = \frac{\xi\,(l - \xi)}{k\,l}$$

Once again, the parts of the bar between $x = 0$ and $x = \xi$ will be displaced by a proportional amount:

$$u(x) = \frac{x\,(l - \xi)}{k\,l}\,,\ 0 \le x \le \xi$$

The portions below $x = \xi$ will also be displaced downward by a proportional amount:

$$u(x) = \frac{\xi\,(l - x)}{k\,l}\,,\ \xi \le x \le l$$

This now gives us a Green's function for this version of the problem:

$$G(x\,;\,\xi) = \begin{cases} \dfrac{x\,(l - \xi)}{k\,l} & x < \xi \\ \dfrac{\xi\,(l - x)}{k\,l} & x \ge \xi \end{cases}$$

Recasting this as an integral

$$G(x\,;\,\xi) = \begin{cases} \int_0^x (l\text{-}\xi)/(l\,k)\,\mathrm{d}\,z & x < \xi \\ \int_x^l \xi/(k\,l)\,\mathrm{d}\,z & x \geq \xi \end{cases}$$

even allows us to make an educated guess about the form of the Green's function for the more general problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f(x)$$

$$u(0) = u(l) = 0$$

that we found so difficult to solve earlier:

$$G(x\,;\,\xi) = \begin{cases} \int_0^x (l\text{-}\xi)/(l\,k(z))\,\mathrm{d}\,z & x < \xi \\ \int_x^l \xi/(k(z)\,l)\,\mathrm{d}\,z & x \geq \xi \end{cases}$$