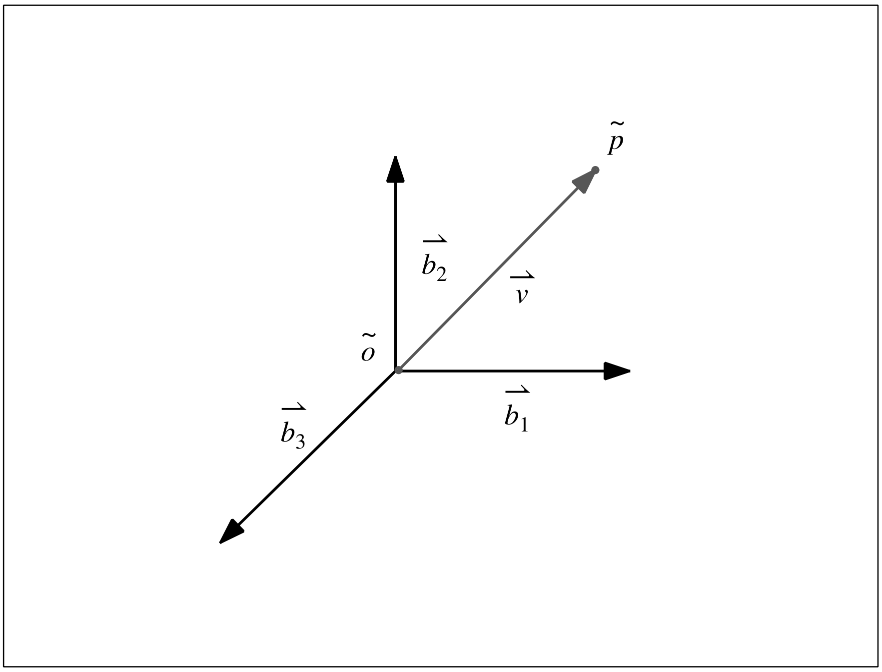
Points

In an earlier lecture you saw how we will be representing and transforming vectors. Vectors are important in computer graphics, but the ultimate objects of interest in graphics are points: our main interest will be in polygons, and a polygon is essentially a set of edges connecting points in space. Those points, the vertices of the polygon, will be the primary objects that we will be working with.

The most common way to represent a point in space is to describe it in terms of a displacement from an origin point:

$$\tilde{p} = \vec{v} + \tilde{o}$$

Here $\tilde{o}$ is an *origin point*, and $\vec{v}$ is a *displacement vector*.



As is our custom, we will write the displacement vector as a linear combination of basis vectors:

$$\tilde{p} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3 + \tilde{o}$$

Written this way, we can say that the coefficients c_1, c_2, and c_3 are the *coordinates* of the point $\tilde{p}$ in our local coordinate system.

Homogeneous coordinates

One slightly awkward aspect of the expression

$$c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3 + \tilde{o}$$

has to do with the fact that it mixes vectors and points. Mathematically, these are different objects, and we have to exercise some care when combining them. For example, it is legal to add two vectors:

$$\vec{g} = \vec{e} + \vec{f}$$

It is also legal to add a vector to a point:

$$\tilde{q} = \tilde{p} + \vec{v}$$

However, it is not legal to add two points:

$$?? \quad \tilde{p} = \tilde{s} + \tilde{t} \quad ??$$

A further complication comes from the fact that we often will want to represent both vectors and points using coordinates. That is, we will want to render the expression

$$c_1 \vec{b_1} + c_2 \vec{b_2} + c_3 \vec{b_3} + \tilde{o}$$

in coordinate form something like this:

$$c_1 \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ b_{3,1} \end{bmatrix} + c_2 \begin{bmatrix} b_{1,2} \\ b_{2,2} \\ b_{3,2} \end{bmatrix} + c_3 \begin{bmatrix} b_{1,3} \\ b_{2,3} \\ b_{3,3} \end{bmatrix} + \begin{bmatrix} o_1 \\ o_2 \\ o_3 \end{bmatrix}$$

This is potentially very confusing, because when written out this way we will have a hard time telling the difference between a vector and a point.

The solution for this confusion is to use a modified system of coordinates known as *homogeneous coordinates*. In this system each object gets an extra fourth coordinate that serves to identify it as a point or a vector. All vectors get 0 as their extra fourth coordinate, while points get a 1 as their extra fourth coordinate.

$$c_1 \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} b_{1,2} \\ b_{2,2} \\ b_{3,2} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} b_{1,3} \\ b_{2,3} \\ b_{3,3} \\ 0 \end{bmatrix} + \begin{bmatrix} o_1 \\ o_2 \\ o_3 \\ 1 \end{bmatrix}$$

One of the attractions of this approach is that it allows us to automatically enforce rules concerning what can be added. For example, adding two vectors works fine:

$$\vec{g} = \vec{e} + \vec{f} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ 0 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \end{bmatrix} = \begin{bmatrix} e_1 + f_1 \\ e_2 + f_2 \\ e_3 + f_3 \\ 0 \end{bmatrix}$$

Adding a vector to a point works,

$$\tilde{q} = \tilde{p} + \vec{v} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} p_1 + v_1 \\ p_2 + v_2 \\ p_3 + v_3 \\ 1 \end{bmatrix}$$

Adding two points lets you know immediately that you have done something wrong:

$$\tilde{s} + \tilde{t} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ 1 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 + t_1 \\ s_2 + t_2 \\ s_3 + t_3 \\ 2 \end{bmatrix}$$

Coordinate Frames

To describe the location of a point we are going to use a *coordinate frame*, which is a combination of an origin point and a set of basis vectors. Any point of interest can be expressed as the sum of the origin point and a displacement vector.

$$\tilde{p} = \vec{v} + \tilde{o}$$

$$= c_1 \vec{b_1} + c_2 \vec{b_2} + c_3 \vec{b_3} + \tilde{o}$$

$$= c_1 \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} b_{1,2} \\ b_{2,2} \\ b_{3,2} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} b_{1,3} \\ b_{2,3} \\ b_{3,3} \\ 0 \end{bmatrix} + \begin{bmatrix} o_1 \\ o_2 \\ o_3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & o_1 \\ b_{2,1} & b_{2,2} & b_{2,3} & o_2 \\ b_{3,1} & b_{3,2} & b_{3,3} & o_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

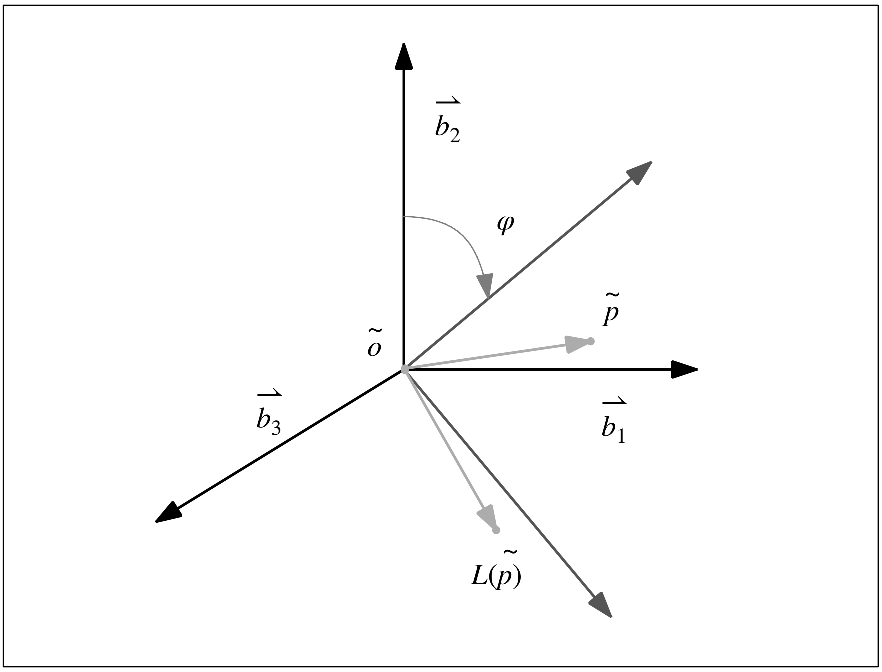
$$= \vec{\mathbf{f}}^{t} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$= \vec{\mathbf{f}}^{t} \mathbf{c}$$

The final vector on the right is the *homogeneous coordinate vector* of the point in the given coordinate frame.

Linear Transformations

Linear transformations in a particular coordinate frame map displacement vectors to new displacement vectors, effectively moving points around.



The point $\tilde{p}$ gets moved to a new point $L(\tilde{p})$ via the motion of its displacement vector:

$$\tilde{p} = \vec{v} + \tilde{o}$$

$$L(\tilde{p}) = L(\vec{v}) + \tilde{o}$$

If we know the matrix representation of the linear transformation, we can express this transformation in homogeneous coordinate form. For example, the transformation shown here is the transformation we looked at in chapter 2. That transformation had a matrix representation

$$L(\vec{v}) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

In homogeneous coordinates this looks like

$$L(\tilde{p}) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

The matrix

$$\begin{bmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is the matrix representation of this linear transformation in homogeneous coordinates.

Affine transformations

In linear algebra, a linear transformation is a transformation that can be represented via a matrix multiplication.

$$L(\vec{v}) = M\,\vec{v}$$

An *affine transformation* is a combination of a linear transformation and a displacement.

$$A(\vec{v}) = M\,\vec{v} + \vec{d}$$

Affine transformations can be extended to points in much the same way that linear transformations can be extended to points. The affine transformation acts on the displacement vector:

$$A(\tilde{p}) = A(\vec{v}) + \tilde{o}$$

$$= M\,\vec{v} + \vec{d} + \tilde{o}$$

In homogeneous coordinates an affine transformation can be represented as a single 4 by 4 matrix.

$$A(\tilde{p}) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} & d_1 \\ M_{2,1} & M_{2,2} & M_{2,3} & d_2 \\ M_{3,1} & M_{3,2} & M_{3,3} & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

How to interpret multiple transformations

Given a transformation

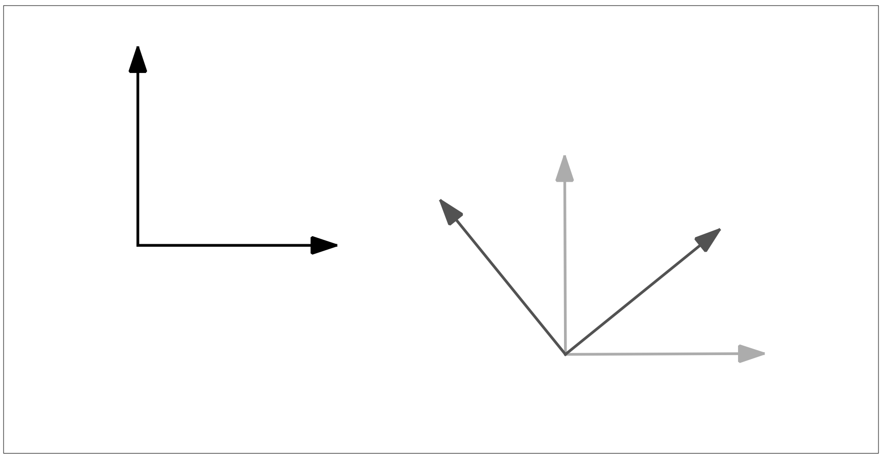
$$\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t\, T\, R$$

How do we interpret the transformation?

Option 1: interpret it as

$$\left(\vec{\mathbf{f}}^t\, T\right) R$$

which is a rotation of the translated frame.



Option 2: interpret is as

$$\overrightarrow{\mathbf{f}}^{t}\,(T\,R)$$

which is a rotation with respect to the original frame, followed by a translation with respect to the original frame.