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MEETING TIMES: MWF 8:30 AM, Y-115. On conference days, half the class will meet at some other time and place, to be decided by the end of week 1.

OFFICE HOURS: 7:00–9:00 PM most Sundays; 1:30–3:00 PM Mondays; 9:00–10:30 AM Thursdays; by appointment.

TEXT: The required textbook for this course is *Mathematical Methods for Physicists* (Fifth Edition) written by George Arfken and Hans Weber and published by Academic Press (2001; ISBN 0-12-059825-6). When you discover that it has 1112 pages, you will understand that we will not in any way cover the material in its entirety (though you already know more than a little of it). As a reference and standard of coverage, this book is unsurpassed; you will appreciate having it in your library. You may also find it valuable to refer occasionally to *Basic Training in Mathematics: A Fitness Program for Science Students* by R. Shankar and published by Plenum (1995; ISBN 0-306-45036-4).

SUPPLEMENTARY MATERIALS: The following additional texts have been placed on reserve in the main library (*) or are in Y104/138(\$\$) for your use.

- *\$ R. Shankar, *Basic Training in Mathematics: A Fitness Program for Science Students* (Plenum Press, New York, 1995) [QA300.S4315]
- * M. L. Boas, *Mathematical Methods in the Physical Sciences*, John Wiley, New York, 1966) [QA37/B725]
- * R. V. Churchill, *Fourier Series and Boundary Value Problems* (McGraw-Hill, New York, 1963) Second Edition [QA404/C6]
- *\$ R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Volume I (Interscience Publishers, New York, 1953) [QA401/C724]
- *\$ H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry* (D. Van-
Nostrand, Princeton, 1956) Second Edition [QA37/M33]
- * J. Mathews and R. L. Walker, *Mathematical Methods of Physics*, (W. A. Benjamin,
Inc., New York, 1970) Second Edition [Library has first edition, QA401/M42.]
- *\$ P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New
York, 1953), Volumes I and II [QC20/M6]
- * I. S. Sokolnikoff and R. M. Redheffer, *Mathematics of Physics and Modern Engineering*
(McGraw-Hill, New York, 1966) Second Edition [QA401/S64]
- \$ G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1962) [517.35/W33t]

SCHEDULE: Physics 440 meets three times a week. About two-thirds of the classes will be lectures (with, of course, opportunity for questions along the way). Seven of the classes will be conference meetings, which will provide opportunity first for student presentation of problem solutions and second for discussion of current material. The accompanying schedule indicates the activity of each day.

READING ASSIGNMENTS: To help define the material to be covered in each class, reading assignments have been placed in the schedule and provided for each class. You are urged to make a first pass at each reading assignment *prior* to the corresponding class.

PROBLEMS: For each class, the schedule also identifies several problems that you should attempt as you study the material for that day. Most likely, you will attempt these problems

after the class rather than before. Most of the problems are at the end of one or another section in Arfken and are identified with a symbol like 8.3.5—the fifth problem at the end of Section 8.3. Some are printed on the last pages of this fact sheet and are identified with a symbol like S16—the sixteenth supplementary problem. In each list of problems, those marked with an asterisk are to be handed in at the class session following the next conference. *Solutions are to be carefully written, well organized, and adequately narrated.*¹ (I reserve the right to return solutions that are unsatisfactory in this regard for rewriting.) In the interests of your keeping abreast of the course and of my efficient grading, I ask that you turn in at each due date whatever of the assignment you have completed; do not withhold an entire assignment solely because you have not completed some of the problems.

CONFERENCE SESSIONS: No new material will be presented in the conference meetings. Instead, each will focus on the material covered by the lectures and reading assignments since the previous conference and on the problems related to that material. These sessions provide an opportunity to ask questions about the material being covered. In addition, students will regularly be asked to present solutions to previously designated problems. On conference days, the class will be divided into two groups, one of which will meet at the regular class time and the other of which will meet at a time convenient to the members of the group. Each group will have only half a dozen students, so each student should anticipate being asked to make a presentation at least once every other conference session.

COMPUTER: The facilities of Lawrence's Computational Physics Laboratory (CPL) will be used regularly during this course. If you haven't already begun to use IDL and MAPLE especially, you should work your way through some of the instructional materials you will find in the CPL, including chapters from *Computation and Problem Solving in Undergraduate Physics (CPSUP)*. More specifically you might want to look at

- The *Lawrence Local Guide* for orientation to the system as a whole.
- Chapter 2 in *CPSUP* for IDL.
- Chapter 5 in *CPSUP* for MAPLE.
- Appendix A in *CPSUP* for L^AT_EX.

Occasionally, assignments will direct you to other chapters in *CPSUP*. Some familiarity with the system, with IDL, and with MAPLE will be assumed. At times, there will be specific direction about tasks that might be accomplished in the CPL; more often, you will be expected to make use of the CPL when in your judgment such use might be helpful.

EXAMINATIONS: There will be two mid-term examinations and a final examination in this course. All will be closed-book examinations.

GRADING: Your grade in this course will be determined largely by your performance on the examinations (30% each hour examination; 40% final examination) and by your diligence and success in working the assigned problems. In particular, a final grade of A will be recorded only for those students who do exceptionally well on all three examinations and who *in addition* complete a substantial fraction (at least 90%) of the assigned problems successfully. Failure to complete some reasonable fraction (say 70-80%) of the assigned problems may depress whatever grade is earned on the examinations alone by as much as a full letter grade.

¹I encourage you to continue developing your skills with L^AT_EX.

HONOR CODE: In Physics 440, you will be expected to submit only your own work on the hour examinations and on the final examination. On the problem assignments, in contrast, you are encouraged to learn from one another by talking together about the problems. If, however, you find yourself more often on the receiving than the contributing end in such interactions, beware; you will ultimately be expected to exhibit your own abilities. By the time you have completed each assignment, you should be absolutely certain you could do each problem on your own should the need arise. The honor pledge should be signed on all submitted work.

Physics 440

SCHEDULE

Spring Term, 2005

All identified problems are to be attempted; problems marked with an asterisk are to be written up neatly and carefully and handed in at the class session after the most immediately following conference. I urge you—at least once in awhile—to use L^AT_EX to prepare your submitted work.

Mo	28	Mar	Partial Differential Equations of Mathematical Physics <i>Read:</i> Your notes; Arfken, Ch 8, pp. 487–496 <i>Problems:</i> S1, *S2, S3, S4, *S6
We	30	Mar	Separation of Variables in Cartesian Coordinates <i>Read:</i> Your notes; Arfken, Ch 8, pp. 506–508 <i>Problems:</i> Arfken, *8.3.5†; S7, S10, *S11 ‡ Find <i>all</i> values of E.
Fr	1	Apr	Fourier Series <i>Read:</i> Arfken, Chapter 14, pp. 863–881; <i>CPSUP</i> Section 11.3.7; Section 5.4.2 of <i>MAPLE Reference Manual</i> , Vs. 14 <i>Problems:</i> Arfken, *14.2.2, *14.3.4†, 14.3.10, *14.4.2(b)‡, *14.4.9; *S12, S13 ‡ Use IDL to plot the series for various numbers of terms. ‡ $\zeta(p) = \sum_{n=1}^{\infty} (1/n^p)$.
Mo	4	Apr	Fourier Series (continued)/The Discrete Fourier Transform <i>Read:</i> Arfken, Chapter 14, pp. 886–901 <i>Problems:</i> Arfken, *14.1.3, *14.3.14†, 14.6.3, *14.6.4 ‡ Use IDL to plot the series for various numbers of terms.
We	6	Apr	CONFERENCE
Fr	8	Apr	ASSIGNMENT 1 DUE Separation of Variables in Other Coordinates <i>Read:</i> Arfken, Ch 8, pp. 508–518 <i>Problems:</i> Arfken, *14.3.6, *8.3.9, 8.4.2; *S16, *S17

Mo	11	Apr	Second-Order Ordinary Differential Equations; Power Series Solutions <i>Read:</i> Arfken, Ch 8, pp. 518–548; Shankar, Section 10.4 <i>Problems:</i> Arfken, *8.5.5, 8.5.12, *8.5.16, 8.6.10, *8.6.11, *8.6.19 [†] [†] Verify explicitly that your solution satisfies the original ODE.
We	13	Apr	Sturm-Liouville Theory; General Orthogonal Functions <i>Read:</i> Arfken, Ch 9, pp. 575–609 (skim the rest of Ch 9 if you wish) <i>Problems:</i> Arfken, 9.1.1, *9.1.8, *9.2.5, 9.3.2, *9.3.3
Fr	15	Apr	CONFERENCE
Mo	18	Apr	ASSIGNMENT 2 DUE Bessel Functions <i>Read:</i> Arfken, Ch 11, pp. 669–702; Shankar, Sections 10.4 and 10.5 <i>Problems:</i> Arfken, 11.1.4, *11.1.25, 11.2.3, *11.2.6(b) [†] , 11.2.9 [†] The answer in Problem 11.2.3 may be useful. DMC will be out of town from immediately after class on 4/18 until bedtime on Tuesday 4/19.
We	20	Apr	Bessel Functions (continued) <i>Read:</i> Arfken, Ch 11, pp. 702–704, 708–712, 722–730 <i>Problems:</i> Arfken, 11.5.1, *11.5.2, *11.5.12(a), 11.7.4, *11.7.13, *11.7.14
Fr	22	Apr	CONFERENCE
Mo	25	Apr	ASSIGNMENT 3 DUE HOUR EXAMINATION 1 (covering Assignments 1, 2, 3)
We	27	Apr	Special Functions (Gamma, Beta, Elliptic, . . .) <i>Read:</i> Arfken, Sections 5.8, 10.1, 10.3, 10.4, 12.1, 12.2, 12.3, 13.1, Appendix 2; skim the rest of Ch 12 and 13. <i>Problems:</i> Arfken *5.8.1, 10.1.5, *10.1.11, *10.1.14, 10.3.6, *10.4.2, 10.4.3, 12.1.1 [†] , *12.2.2 [†] Include in your solutions sketches or computer produced graphs of (a) the potential along the z axis, (b) the potential along the illustrated axis perpendicular to the z axis, and (c), as a surface plot, the potential in the plane of the paper.
Fr	29	Apr	General Orthogonal Coordinates <i>Read:</i> Arfken, Chapter 2, pp. 103–126 <i>Problems:</i> Arfken, *2.1.3, *2.2.2, 2.4.5, *2.4.7, 2.4.11, *2.5.3
Mo	2	May	CONFERENCE
We	4	May	ASSIGNMENT 4 DUE Functions of a Complex Variable <i>Read:</i> Arfken, Chapter 6, pp. 389–401; Shankar, Chapter 5 and Chapter 6, pp. 107–111 <i>Problems:</i> Arfken, *6.1.8, 6.1.10, 6.2.5, *6.2.6, *6.2.8
Fr	6	May	MID-TERM READING PERIOD – NO CLASS; GALA OPEN HOUSE
Sa	7	May	INSTALLATION OF PRESIDENT JILL BECK

Mo	9	May	Integral Theorems <i>Read:</i> Arfken, Chapter 6, pp. 404–415 <i>Problems:</i> Arfken, 6.3.2, *6.3.3, 6.4.1, *6.4.4
We	11	May	CONFERENCE
Fr	13	May	ASSIGNMENT 5 DUE The Laurent Series; Mapping <i>Read:</i> Arfken, Chapter 6, pp. 416–433 <i>Problems:</i> Arfken, *6.5.2, 6.5.5, *6.6.1, *6.6.3(a), 6.7.3
Mo	16	May	The Calculus of Residues <i>Read:</i> Arfken, Chapter 7, pp. 439–462; Shankar, Section 6.4 <i>Problems:</i> Arfken, 7.1.1, *7.1.2, *7.2.5, 7.2.14, *7.2.21
We	18	May	CONFERENCE
Fr	20	May	ASSIGNMENT 6 DUE HOUR EXAMINATION 2 (covering Assignments 4, 5, 6)
Mo	23	May	Fourier Transforms <i>Read:</i> Arfken, Chapter 15, pp. 905–926 <i>Problems:</i> Arfken 15.1.3, 15.3.2, *15.3.3, *15.3.4, *15.3.10, 15.3.16, *15.4.3
We	25	May	Green's Functions <i>Read:</i> Arfken, Ch 1, pp. 84–90; Ch 8, pp. 548–563; Ch 9, pp. 616–626; Shankar, Section 10.6 <i>Problems:</i> Arfken, 1.15.3, *1.15.4, 1.15.9, *8.7.3, *9.5.2(a)
Fr	27	May	Neat Solutions to the Wave and Heat Flow Equations <i>Read:</i> Take good notes, but you might also find Sokolnikoff and Redheffer, Second Edition, Chapter 7, Sections 3, 17, and 19 to be useful. <i>Problems:</i> *S33, S34, *S35, S36
Mo	30	May	MEMORIAL DAY HOLIDAY
We	1	Jun	Group Theory and Symmetry Arfken, Chapter 4, pp. 237–241, 286–296 <i>Problems:</i> Arfken *4.1.3, *4.7.3, 4.7.5, 4.7.11, 4.7.12, *4.7.18
Fr	3	Jun	CONFERENCE
Sa	4	Jun	(noon) ASSIGNMENT 7 DUE
We	8	Jun	8:30 AM FINAL EXAMINATION (covering entire course)

1. The vertical displacement $u(x, t)$ of a stretched string satisfies the inhomogeneous wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} = \tau \frac{\partial^2 u}{\partial x^2} - \rho g$$

Suppose the string has length l and is fixed at both ends [$u(0, t) = 0$; $u(l, t) = 0$].

- Find the shape of a uniform string (ρ constant) hanging at rest [$u(x, t)$ does not depend on t] under its own weight.
 - Sketch a graph of that shape.
 - Find an expression giving the sag—deflection at the center—as a function of τ and sketch a graph of sag versus tension.
2. The temperature $u(x, t)$ in one-dimensional heat flow—think heat flow through a wall oriented perpendicular to the x axis and having thickness h —satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$$

where $\alpha = K/\rho c$ (K = thermal conductivity; ρ = mass density of material; c = heat capacity per unit mass). Suppose that the temperature is maintained at the values $u(0, t) = u_0$ and $u(h, t) = u_h$, and consider only the steady state temperature distribution [$u(x, t)$ depends only on x].

- If the material between $x = 0$ and $x = h$ is uniform (has the same thermal properties throughout), (i) find an expression for the steady state temperature $u(x)$ in the wall, (ii) sketch a graph of $u(x)$ versus x , and (iii), remembering that the rate of heat flow in the positive x direction at any point is given by $-KA(\partial u/\partial x)$, find an expression for that rate of heat flow and argue that heat flows from the warmer to the cooler side of the wall.
 - Suppose the wall is made up of two *different* materials, with different values of α . To be more specific, suppose that material characterized by K_1, c_1, ρ_1 fills the region $0 < x < \frac{1}{2}h$ and that material characterized by K_2, c_2, ρ_2 fills the region $\frac{1}{2}h < x < h$. (i) Find the steady state temperature in this composite wall, (ii) sketch a graph of this temperature distribution, and (iii) examine the way that distribution varies with changes in the characteristics of the two materials. *Hints:* (1) Solve the heat-flow equation in the two regions of x and impose on the solutions not only the required temperatures u_0 and u_h at the outside faces of the wall but also the requirement that both the temperature $u(x)$ and the rate of heat flow $-KA(du/dx)$ be continuous across the interface between the two different materials. (2) Seek a single parameter—some product and/or ratio of the physical quantities—that all by itself embodies the essential dependence of the graph on the physical properties of the two materials and then plot temperature distributions for several different values of that parameter.
3. The time-dependent quantum wave function satisfies the equation

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi = -i\hbar \frac{\partial \Psi}{\partial t}$$

Suppose that the wave function has the form $\Psi(\mathbf{r}, t) = \psi(\mathbf{r})e^{i\omega t}$. Find the time-independent equation satisfied by the function $\psi(\mathbf{r})$. Remember that, in quantum mechanics, a state having the specified time dependence is a state of definite energy E and $E = \hbar\omega$.

4. [Sokolnikoff and Redheffer, pg. 439, #4] (a) By computing u_x, u_y , and u_{xy} , obtain a second-order partial differential equation for $u(x, y) = f(x)g(y)$. (b) Show that your result is equivalent to $(\ln u)_{xy} = 0$.

5. Find a second-order partial differential equation satisfied by the function $u(x, y) = \ln(x^2 + y^2)$.
6. By using the chain rule, show that the function $u(x, t) = Af(x - ct) + Bg(x + ct)$ satisfies the wave equation $u_{xx} = u_{tt}/c^2$, where A , B , and c are constants and $f()$ and $g()$ are arbitrary functions.
7. [Sokolnikoff and Redheffer, pg. 440, #11] Some partial differential equations can be treated as ordinary differential equations, except that the “constants” of integration depend on the variables not involved in the differentiation. Solve the following equations for $u(x, y)$:

$$\begin{array}{llll} \text{(a)} & u_x = 2x & \text{(b)} & u_y = x \\ \text{(c)} & u_{xx} = 0 & \text{(d)} & u_{xx} = y \\ \text{(e)} & u_x = 2u & \text{(f)} & u_{yy} = u \\ \text{(g)} & u_{xx} + u = x + y & \text{(h)} & u_{xy} + u_x = 1 \end{array}$$

8. Find a general solution (involving two arbitrary functions) for the equation $u_{xy} + u_x = 1$.
9. Find a function $u(x, y)$ satisfying the following conditions:

$$\begin{array}{l} \text{(a)} \quad u_{xx} = 6xy; \quad u(0, y) = y, \quad u_x(1, y) = 0. \\ \text{(b)} \quad u_{xy} = 4x; \quad u(0, y) = y, \quad u(x, 0) = \sin(x). \\ \text{(c)} \quad u_{xy} = 2x \sin(3y); \quad u(1, y) = y, \quad u(x, 0) = 0. \end{array}$$

10. By applying the technique of separation of variables to the 1D wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

- (a) find general product-form solutions $X_\lambda(x) T_\lambda(t)$ for all values—positive, negative, zero—of the separation constant λ and (b), by imposing the boundary and initial conditions

$$u(0, t) = u(a, t) = 0 \quad ; \quad u(x, 0) = f(x) \quad ; \quad \frac{\partial u}{\partial t}(x, 0) = g(x)$$

which tie the string down at its two ends and stipulate arbitrary initial displacement and velocity over the interval $0 < x < a$, show that

$$u(x, t) = \sum_{n=1}^{\infty} \left[C_n \sin \frac{n\pi ct}{a} + D_n \cos \frac{n\pi ct}{a} \right] \sin \frac{n\pi x}{a}$$

where C_n and D_n must be chosen so that

$$f(x) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{a} \quad ; \quad g(x) = \sum_{n=1}^{\infty} \frac{n\pi c C_n}{a} \sin \frac{n\pi x}{a}$$

11. By applying the technique of separation of variables to the 1D diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$$

- (a) find general product-form solutions $X_\lambda(x) T_\lambda(t)$ for all values—positive, negative, zero—of the separation constant λ and (b), by imposing the boundary and initial conditions

$$u(0, t) = u_1 \quad ; \quad u(l, t) = u_2 \quad ; \quad u(x, 0) = f(x)$$

which fix the temperature at the two ends and stipulate arbitrary initial temperature in the interval $0 < x < l$, show that

$$u(x, t) = \frac{u_2 - u_1}{l} x + u_1 + \sum_{n=1}^{\infty} A_n e^{-n^2 \pi^2 \alpha^2 t / l^2} \sin \frac{n\pi x}{l}$$

where A_n must be chosen to that

$$f(x) - \frac{u_2 - u_1}{l}x - u_1 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

Hint: Note that the given solution is a superposition—remember that the equation is linear—of two solutions to the partial differential equation. The terms *outside* the sum constitute one solution satisfying the given boundary conditions and the requirement of time independence; they express the steady-state temperature distribution attained after a long time. The sum constitutes the second solution; since it satisfies the boundary conditions $u(0, t) = u(l, t) = 0$, the total solution (first *plus* second pieces) still satisfies the original boundary conditions. The initial condition is imposed on the *total* solution, not on either piece separately.

12. The small-amplitude vertical motion $u(x, y, t)$ of a rectangular membrane (drum head) lying in the horizontal xy plane satisfies the 2D wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

Take the sides of the drum head to be located in the xy plane at $x = 0$, $x = a$, $y = 0$ and $y = b$ so that the condition that the drumhead be “tied down” at its perimeter is expressed mathematically by the requirement that $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$. (a) Assuming a sinusoidal solution of the form $u(x, y, t) = U(x, y) \cos \omega t$ and applying separation of variables to the partial differential equation satisfied by $U(x, y)$, show that the natural frequencies of oscillation of this drumhead are given by

$$\omega_{mn} = \pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

where $m, n = 1, 2, 3, \dots$ (b) Find the functions $U_{mn}(x, y)$ associated with these eigenvalues and, using `surface`, `shade_surf`, and/or `contour` in IDL, obtain graphs showing the initial shape of the membrane for some of the lower frequency modes. *Optional:* Again using IDL, produce an animated display of the motion of some of the lower frequency modes.

13. (a) Find a *cosine* series that converges to the function $f(x) = 1$ when $0 < x < \pi$ and to the function $f(x) = 0$ when $\pi < x < 2\pi$. (b) Sketch a graph of the function represented by that series over the interval $-6\pi < x < 6\pi$. (c) Use IDL to plot graphs of the truncated series for various numbers of included terms. (d) State the value to which the series converges at $x = 0$ and use that value to deduce that

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}$$

14. [Sokolnikoff and Redheffer, pg. 62, #3] Find the Fourier series for the function $f(x) = x + x^2$ over the interval $-\pi < x < \pi$ and then use Parseval’s theorem to deduce the interesting value

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

for the ζ function of argument 2.

15. [Sokolnikoff and Redheffer, pg. 66, #7] Find the Fourier series representing the function $f(x) = \cos(ax)$ in the interval $-\pi < x < \pi$, assuming that a is *not* an integer. Argue that the series converges to the value $\cos(a\pi)$ when $x = \pi$, and deduce the interesting result

$$\pi a \cot(\pi a) = 1 - \sum_{n=1}^{\infty} \frac{2a^2}{n^2 - a^2}$$

16. [Arfken, problem 8.3.6, with typos corrected] For a homogeneous spherical solid with constant thermal diffusivity K and no heat sources, the equation of heat conduction becomes

$$\frac{\partial u(r, t)}{\partial t} = K \nabla^2 u(r, t)$$

where u is assumed to depend only on the spherical radial coordinate r and the time t . (a) Assume a solution of the form $u(r, t) = R(r)T(t)$ and apply separation of variables to show that the radial function $R(r)$ satisfies

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + \alpha^2 r^2 R = 0$$

(b) Transform the independent variable to $x = \alpha r$ and then transform the dependent variable by introducing $R(x) = y(x)/\sqrt{x}$ to show that this equation reduces to

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left[x^2 - \frac{1}{4} \right] y = 0$$

This equation is the Bessel equation of order $\frac{1}{2}$. The function $y(x)$ is a $1/2$ -order Bessel function; the function $R(x) = y(x)/\sqrt{x}$ is called a $1/2$ -order *spherical* Bessel function

17. Laplace's equation in parabolic coordinates u, v, ϕ is given by

$$\nabla^2 \psi = \frac{1}{(u^2 + v^2)} \left[\frac{1}{u} \frac{\partial}{\partial u} \left(u \frac{\partial \psi}{\partial u} \right) + \frac{1}{v} \frac{\partial}{\partial v} \left(v \frac{\partial \psi}{\partial v} \right) \right] + \frac{1}{u^2 v^2} \frac{\partial^2 \psi}{\partial \phi^2} = 0$$

Assume a solution of the form $\psi(u, v, \phi) = U(u)V(v)\Phi(\phi)$ and apply the method of separation of variables to find equations satisfied by $U(u)$, $V(v)$, and $\Phi(\phi)$. (The quantum treatment of the hydrogen atom has a particularly revealing solution in parabolic coordinates!)

18. Reserved for a problem involving showing what the variable transformation $\xi = x - ct$, $\eta = x + ct$ does to the wave equation and then showing that $u(x, t) = f(\xi) + g(\eta) = f(x - ct) + g(x + ct)$. Maybe work out d'Alembert solution to the equation by imposing on this solution general initial conditions.
19. As used in statistical data analysis, the Gaussian distribution for a variable t is usually expressed in terms of the standard deviation σ , the distribution function being

$$\frac{1}{\sqrt{2\pi} \sigma} e^{-t^2/2\sigma^2}$$

Thus, the probability of finding a value between $-x$ and $+x$ is given by

$$P(-x < t < x) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-x}^{+x} e^{-t^2/2\sigma^2} dt$$

Show analytically that $P(-x < t < x) = \text{erf}(x/(\sqrt{2}\sigma))$ and then use numerical integration to evaluate $P(-\sigma < t < \sigma)$, $P(-2\sigma < t < 2\sigma)$, and $P(-3\sigma < t < 3\sigma)$. Note that, by definition, function $\text{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$. *Optional:* Obtain a graph of $P(-x < t < x)$ versus x for $-4.0 < x < 4.0$.

20. The n -th order Bessel function is defined by the integral

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin \theta) d\theta$$

Using numerical integration, obtain graphs of $J_0(x)$, $J_1(x)$, and $J_2(x)$ over the range $0 \leq x \leq 10$.

21. A circular ring of radius a resides in the xy plane with its center at the origin and carries a charge Q uniformly distributed about its perimeter. The electrostatic potential established by this ring at an observation point whose cylindrical coordinates are (r, ϕ, z) is given by

$$\frac{V(r, \phi, z)}{Q/4\pi\epsilon_0 a} = \frac{1}{\pi} \int_0^\pi \left(1 - 2\frac{r}{a} \cos \phi' + \frac{r^2}{a^2} + \frac{z^2}{a^2} \right)^{-1/2} d\phi'$$

Explore this integral as a function of r/a and z/a . Graphs of the integral as a function of r/a for chosen values of z/a may be especially meaningful.

22. Explore the behavior of the Van der Pol oscillator described in dimensionless form by the equation

$$\frac{d^2 X}{dT^2} = (1 - X^2) \frac{dX}{dT} - X$$

Don't fail to examine the trajectory in phase space (velocity versus position). Convince yourself that the final, steady-state motion is independent of the initial conditions. Solve both with `ludiffeq_23` and with `LSODE`.

23. Examine the behavior of the large-amplitude simple pendulum described in dimensionless form by the equation

$$\frac{d^2 X}{dT^2} = -\sin X$$

where X is the angular displacement *in radians*. Look particularly at the oscillation with various initial angles. Try to learn how the period depends on amplitude. Try starting the pendulum at the bottom ($X(0) = 0$) with several initial angular velocities. How large can the angular velocity be before the pendulum begins to swing over the top?

24. Consider the quantum harmonic oscillator, described in dimensionless terms by the equation

$$\frac{d^2 \psi}{dx^2} + (2\mu - x^2)\psi = 0$$

where μ is a parameter in the equation related to the energy by $E = \mu\hbar\omega$. While the differential equation has solutions for any value of μ , only those solutions for which $\mu = n + \frac{1}{2}$ are physically acceptable. (Only those solutions decay to zero for large x .) Remember that $n = 0, 2, 4, \dots$ yields even solutions (for which one can suppose $\psi(0) = 1$ and $d\psi(0)/dx = 0$) and $n = 1, 3, 5, \dots$ yields odd solutions (for which one can suppose $\psi(0) = 0$ and $d\psi(0)/dx = 1$). Using IDL's routine `ludiffeq_23`, obtain graphs of the wave functions for several n , including some as large as $n = 20$. Consider only $x > 0$; the negative half can be obtained by symmetry.

25. Use `LSODE` to study the approach of the temperature in a one-dimensional rod to equilibrium when the end at $x = 1$ is insulated and the end at $x = 0$ is maintained at a temperature of 100 °C. Let the entire rod be initially at 0 °C. The temperature satisfies the (dimensionless) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

One way to display the results is to show a family of graphs of $u(x, t)$ versus x for selected t . Alternatively, a graph of u as a surface over the xt plane might be interesting. Perhaps you should try to do both.

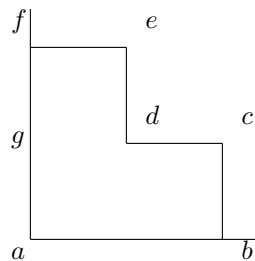
26. Use `LSODE` to study the motion of a string extending over the interval $0 \leq x \leq l$ if the string is tied down at both ends and is displaced initially into a triangular shape by drawing the string aside an amount $0.1l$ at its center and releasing the string from rest. The displacement of the string satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

where v is the speed of propagation of waves on the string. Begin by casting the problem in a dimensionless form. Repeat if the string is initially at rest but is given the initial velocity

$$\begin{aligned} u_t(x, 0) &= 0 & 0 \leq x \leq 0.2l, 0.4l \leq x \leq l \\ &= 1 & 0.2l < x < 0.4l \end{aligned}$$

27. A two-dimensional plate is made by taking a square of side 1 unit and cutting a square of side 0.5 unit out of one corner as shown in the figure below. Suppose the temperature along sides af and ab is maintained at 0°C , the temperature increases linearly from 50°C at e and at c to 100°C at d , and sides fe and bc are insulated. If the entire plate is initially at 0°C , use LSODE to study the approach of the temperature to equilibrium and, in particular, determine the equilibrium temperature. The temperature u in this region satisfies the (dimensionless) diffusion equation $\nabla^2 u = u_t$. *Hint:* Set up a Cartesian coordinate system in which the origin lies at a , the side ab lies along the x axis, and the side af lies along the y axis.



28. Suppose a thin membrane stretched over the frame shown in the previous problem is drawn aside from the plane of the paper so that the rectangle $gdcb$ is undisplaced and, in a Cartesian coordinate system with its origin at a and the lines ab and af along the x and y axes, respectively, the square $gdfe$ assumes a shape given by $u(x, y, 0) = 64x(\frac{1}{2} - x)(y - \frac{1}{2})(1 - y)$. That membrane is then released from rest. Use LSODE to study the subsequent motion of the membrane. In particular, obtain surface plots of the shape of the membrane over the xy plane for enough different times to make the character of the motion clear. Remember that, in a dimensionless presentation, the motion of the membrane satisfies the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

29. Find a solution to Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for the steady-state temperature in the interior of the unit square, imposing on your solution the boundary conditions

$$u_y(x, 0) = 0 \quad ; \quad u_x(1, y) = 0 \quad ; \quad u(0, y) = 50y \quad ; \quad u(x, 1) = 50 + 50x$$

That is, the lower and right edges are insulated, the temperature rises linearly from 0°C to 50°C along the left edge, and then from 50°C to 100°C along the upper edge. Use full discretization of the differential equation and incorporate the boundary conditions to find a set of simultaneous linear equations for the (approximate) solution at the 16 nodes shown in the figure, expressing those equations in the form $Au = b$, where $u = [u_1, u_2, u_3, \dots, u_{15}, u_{16}]^T$. Then use IDL to solve the equations and plot the solution. *Optional:* Solve the same problem on a more refined grid.

y				
13	14	15	16	
9	10	11	12	
5	6	7	8	
1	2	3	4	x

30. A thin membrane is stretched between two concentric and coplanar rings, the inner of radius a and the outer of radius $b = \alpha a$. The transverse motion of this membrane satisfies the 2D wave equation. Since polar coordinates are most convenient, that equation is best written in the form

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

which, for simple harmonic oscillations $u(r, \phi, t) = U(r, \phi) \cos \omega t$ reduces to

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \phi^2} = -\frac{\omega^2}{v^2} U$$

Then, if we assume $U(r, \phi) = R(r) \Phi(\phi)$, we find that

$$\frac{d^2 \Phi}{d\phi^2} + m^2 \Phi = 0 \quad \implies \quad \Phi = A \cos m\phi + B \sin m\phi$$

where m must be an integer (including 0) for solutions periodic with period 2π in ϕ , and $R(r)$ satisfies

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + \left(\frac{\omega^2}{v^2} r^2 - m^2 \right) R = 0$$

You, of course, recognize this equation as the m -th order Bessel equation. Suppose we now measure r in units of a —i.e., set $x = r/a$. The radial equation then becomes

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - \frac{m^2}{x^2} R = -\lambda R$$

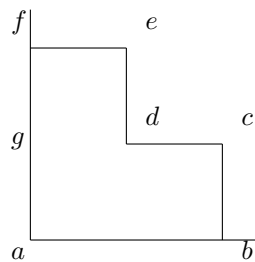
where $\lambda = \omega^2 a^2 / v^2$. This equation is to be solved subject to the requirements that $R(1) = 0$ and that $R(\alpha) = 0$.

Taking $\alpha = 2$, find and plot the lowest three radial wave functions for $m = 0, 1$, and 2 , plot graphs to scale showing the nodal lines in the membrane for all nine of those modes, express the frequency of each mode as a multiple of the lowest frequency, and obtain graphs of $R(x)$ over the interval $1 \leq x \leq 2$ for each mode. *Hint:* Divide the region $1 \leq x \leq 2$ into 10 equal-sized segments, discretize the differential equation, find and then solve a suitable (approximately equivalent) matrix eigenvalue problem. Then, refine your solution by using a finer division of the region $1 \leq x \leq 2$ —say into 100 segments. Note that the equations

$$\frac{d^2 R}{dx^2} \approx \frac{R_{i+1} - 2R_i + R_{i-1}}{\Delta x^2} \quad \text{and} \quad \frac{dR}{dx} \approx \frac{R_{i+1} - R_{i-1}}{2 \Delta x}$$

may provide suitable discretization of the differential equation.

31. In two dimensions, a conducting plate maintained at an electrostatic potential of 100 V occupies the region $-5.0 \leq x \leq 5.0$, $1.0 \leq y \leq 1.5$. A second conducting plate maintained at an electrostatic potential of -100 V occupies the region $-5.0 \leq x \leq 5.0$, $-1.5 \leq y \leq -1.0$. (Length units are arbitrary.) Use MARC/MENTAT to find the electrostatic potential and the electric field in the region around these two plates. *Hints:* The potential, of course, goes to zero at infinite distances from the plates. Infinite distances, however, are hard to represent in computer solutions. Suppose the plates are enclosed in a square region for which $-10.0 \leq x, y \leq 10.0$ and suppose the electrostatic potential to be zero on the perimeter of that square. Thus, the problem is to solve Laplace's equation in the region interior to that square and outside the plates when one plate is held at 100 V, the second plate is held at -100 V, and the boundary of the square is held at 0 V. Give some thought to making the mesh finer in the region within and around the ends of the plates.
32. A two-dimensional plate is made by taking a square of side 1 unit and cutting a square of side 0.5 unit out of one corner as shown in the figure below. Suppose the temperature along sides af and ab is maintained at 0 °C, the temperature increases linearly from 50 °C at e and at c to 100 °C at d , and sides fe and bc are insulated. Use MARC/MENTAT to find the steady state temperature in this plate. The steady-state temperature distribution satisfies Laplace's equation $\nabla^2 u = 0$.



33. [(almost) Sokolnikoff and Redheffer, pg. 449, #6] A stretched infinite string initially in its equilibrium position is struck so that the segment $-a \leq x \leq a$ is given an initial velocity v_0 . (a) Use d'Alembert's formula to find the displacement, and sketch the displacement curves for $t = a/v$ and $t = 2a/v$, where v is the speed of propagation of the wave in the string. (b) Argue that ultimately the string comes to rest along a line parallel to its original position but displaced in the direction of the initial velocity, and find the magnitude of that displacement. *Hint:* Find the velocity of the string at $x = 0$ for $t > a/v_0$.
34. Suppose a string is observed at time zero to have a shape given by $u(x, 0) = f(x)$. A string released from rest with this initial shape will be observed to have two equal pulses, each half the size of the original, one traveling to the left and the other to the right. It is observed, however, that in the case of this problem the *full* pulse propagates to the right—i.e., towards positive x . The initial velocity of the string must therefore be something other than zero. Find the necessary initial velocity $u_t(x, 0)$.
35. Find the d'Alembert form of the solution to the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \quad ; \quad u(x, 0) = f(x) \quad ; \quad u_t(x, 0) = g(x)$$

on $-\infty < x < \infty$, $0 < t < \infty$ by taking a Fourier transform with respect to x , solving the resulting *ordinary* differential equation for $\tilde{u}(k, t)$, imposing the (Fourier transform of) the initial conditions to determine the “constants” in that solution, and finally inverting the transform to return to x, t space.

36. [Vvdensky, pg. 303, #26] Patterning your development after that illustrated in class, show that the solution to the (dimensionless) *inhomogeneous* heat flow equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = q(x, t) \quad ; \quad u(x, 0) = f(x)$$

is expressible as

$$\begin{aligned} u(x, t) &= \int_0^t \int_{-\infty}^{\infty} \frac{q(x', t')}{\sqrt{4\pi(t-t')}} e^{-(x-x')^2/[4(t-t')]} dx' dt' \\ &+ \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(x') e^{-(x-x')^2/[4t]} dx' \end{aligned}$$

Assignments from Previous Years on PDEs Numerically

- We 5 May ASSIGNMENT 4 DUE
The Initial Value Problem for ODEs (`ludiffeq_23`, `LSODE`)
Read: Arfken, pp. 529–533; CPL-323; CPL-363, Sections 1,2;
take good notes. You may also want to be aware
of the program DSS2 (see CPL-100, CPL-101).
Problems: *S22, S23, *S24
- Fr 7 May The Initial Value Problem for PDEs (`LSODE`)
Read: CPL-363, Sections 3–7
Problems: *S25, S26, *S27, S28
- Mo 10 May Boundary Value Problems/Eigenvalues (Finite Difference Methods)
Read: Take good notes.
Problems: *S29, *S30
- We 12 May CONFERENCE
- Fr 14 May ASSIGNMENT 5 DUE
Finite Element Methods in One Dimension
Read: CPL-624
Problems: At the end of CPL-624, problems *2, *3, *6
- Mo 17 May Finite Element Methods in Two Dimensions; MARC/MENTAT
Read: Take good notes; CPL-625, CPL-640
Problems: *S31, S32
- We 19 May CONFERENCE
Symbolic Approaches to ODEs CPL-061, CPL-341
, *SS4, SS5, *SS6, SS7

SS4. Seek solutions to each of the following equations by applying MAPLE's ODE command:

- a. $y'' + by' + ky = 0$
- b. $x^2y'' + xy' + (x^2 - n^2)y = 0$
- c. $x^2y'' + xy' + (k^2x^2 - n^2)y = 0$

Note that (c) can be obtained from (b) with the simple variable change $x \rightarrow kx$. Can you find a way to help MAPLE recognize that fact?

SS5. Try MAPLE's `SERIES` command with the Legendre equation in Arfken, problem 8.5.5. Try it first with n general. Then try it with a couple of specific values of n , one even and the other odd. Remember to issue the commands `LOAD(SINGS)` and `LOAD(SERIES)` before your first use of `SERIES`.

SS6. Try MAPLE's `SERIES` command with the equation in Arfken, problem 8.5.16. Remember to issue the commands `LOAD(SINGS)` and `LOAD(SERIES)` before your first use of `SERIES`.

SS7. Search for a way to persuade MAPLE to do the algebra associated with solving problem 8.5.12 in Arfken. This will probably not be easy. The most productive route may be to assume a series of the form

$$u(\xi) = \xi^{m/2} (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \dots)$$

writing out the series explicitly, truncating it at some point, substituting it into the equation, extracting coefficients, etc. MAPLE should be able to do it with this kind of help from you.

WEEK 6, BEGINNING MONDAY 4 MAY:

- Functions of a Complex Variable
Read: Arfken, Chapter 6, pp. 364–377
Problems: Arfken, *6.1.8, 6.1.10, 6.2.5, *6.2.6, *6.2.8
- Integral Theorems
Read: Arfken, Chapter 6, pp. 377–389
Problems: Arfken, 6.3.2, *6.3.3, 6.4.1, *6.4.4

WEEK 7, BEGINNING MONDAY 11 MAY:

- The Laurent Series; Mapping
Read: Arfken, Chapter 6, pp. 389–409
Problems: Arfken, *6.5.2, 6.5.5, *6.6.1, *6.6.3(a), 6.7.3
- The Calculus of Residues
Read: Arfken, Chapter 7, pp. 410–439
Problems: Arfken, 7.1.1, *7.1.2, *7.2.5, 7.2.14, *7.2.21

WEEK 8, BEGINNING MONDAY 18 MAY:

- General Orthogonal Coordinates
Read: Arfken, Chapter 2, pp. 100–126
Problems: Arfken, *2.1.3, *2.2.2, 2.4.5, *2.4.7, 2.4.11, *2.5.3